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Perturbation Theory Around Nonnested Fermi Surfaces. I. Keeping the Fermi Surface Fixed

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The perturbation expansion for a general class of many-fermion systems with a nonnested, nonspherical Fermi surface is renormalized to all orders. In the limit as the infrared cutoff is removed, the counterterms converge to a finite limit which is differentiable in the band structure. The map from the renormalized to the bare band structure is shown to be locally injective. A new classification of graphs as overlapping or nonoverlapping is given, and improved power counting bounds are derived from it. They imply that the only subgraphs that can generate r factorials in the rth order of the renormalized perturbation series are indeed the ladder graphs and thus give a precise sense to the statement that "ladders are the most divergent diagrams." Our results apply directly to the Hubbard model at any filling except for half-filling. The half-filled Hubbard model is treated in another place.

KEY WORDS: Many-fermion systems; perturbation theory; renormalization nonspherical Fermi surfaces; Hubbard model; overlapping graphs.

CONTENTS

This paper is dedicated to the memory of Ansgar Schnizer.

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1. INTRODUCTION AND OVERVIEW

1.1. The Problem

Consider the following problem in many-body physics. Let Λ be a finite box in d-dimensional space, i.e., $\Lambda \subset \mathbb{R}^d$ or $\Lambda \subset \Gamma$, where Γ is a lattice in $\mathbb{R}^d$, let $c_\sigma(\mathbf{x})$ and $c_\sigma^+(\mathbf{x})$ be fermionic annihilation and creation operators obeying the canonical anticommutation relations $\{c_\sigma(\mathbf{x}), c_{\sigma'}^+(\mathbf{x}')\} = \delta_{\sigma\sigma'}\delta(\mathbf{x}-\mathbf{x}')$, and let $\mathscr{F}$ be the fermionic Fock space generated by this algebra.[1] Let $H_\Lambda = H_0 + \lambda V$ be the operator on $\mathscr{F}$ given by

$$H_0 = \sum_\sigma \int ds(\mathbf{x})\, c_\sigma^+(\mathbf{x})(T+U)\, c_\sigma(\mathbf{x}) \tag{1.1}$$

where T is an operator describing the one-particle kinetic energy, U is multiplication by a periodic potential, and $\int ds(\mathbf{x})$ denotes $\int_\Lambda d\mathbf{x}$ for a continuous system and $\sum_{\mathbf{x}\in\Lambda}$ for a system on a lattice. Let $n_\sigma(\mathbf{x}) = c_\sigma^+(\mathbf{x})\, c_\sigma(\mathbf{x})$ be the number operator at $\mathbf{x}$ for spin σ. The interaction

$$V = \sum_{\sigma,\sigma'} \int ds(\mathbf{x}) \int ds(\mathbf{x}')\, n_\sigma(\mathbf{x})\, v_{\sigma\sigma'}(\mathbf{x}-\mathbf{x}')\, n_{\sigma'}(\mathbf{x}') \tag{1.2}$$

is assumed to be short-range (see Assumption A1 below). The Hamiltonian H_Λ describes many electrons in a crystal or on a lattice that interact with a stationary ionic background through U and with each other through the pair potential V. If the coupling strength of the electron–electron interaction $\lambda = 0$, the electrons move independently according to the one-particle

Schrödinger operator $T + U(x)$. In the continuum system $T = -\Delta/2m$ is the Laplacian and $U(x+y) = U(x)$ for all $y \in \Gamma$, where the lattice Γ is generated by d linearly independent vectors in $\mathbb{R}^d$ (e.g., $\Gamma = \mathbb{Z}^d$); in the case of a lattice system, $U = 0$ and the kinetic energy T is defined by the hopping matrix between the sites of the lattice. For $\lambda \neq 0$, the potential V takes into account interactions such as screened electromagnetic interactions. A slight generalization of (1.2) allows for inclusion of phonon-mediated interactions.

Let $\beta = 1/kT$ be the inverse temperature and define the grand canonical partition function Z_Λ as

$$Z_\Lambda = \operatorname{tr} e^{-\beta(H_\Lambda - \mu N_\Lambda)} \tag{1.3}$$

where

$$N_\Lambda = \sum_\sigma \int_\Lambda ds(\mathbf{x})\, n_\sigma(\mathbf{x}) \tag{1.4}$$

is the number operator, μ is the chemical potential, and the trace is over Fock space. For an observable $\mathcal{O}$, i.e., a polynomial in the fermion operators, the thermal expectation value is defined as

$$\langle \mathcal{O} \rangle_\Lambda = \frac{1}{Z_\Lambda} \operatorname{tr}(e^{-\beta(H_\Lambda - \mu N_\Lambda)} \mathcal{O}) \tag{1.5}$$

The question we are interested in is whether the thermodynamic limit $\mathcal{G} = \lim_{\Lambda \to \infty} \mathcal{G}_\Lambda$ of the connected Green functions $\mathcal{G}_\Lambda = \langle c^+_{\sigma_1}(\mathbf{x}_1) \ldots c_{\sigma'_m}(\mathbf{x}'_m) \rangle_{\Lambda,\mathrm{conn}}$, which are special cases of $\mathcal{O}$ above, exists and whether in finite volume a weak-coupling expansion

$$\mathcal{G} = \sum_{r=0}^{\infty} \lambda^r G_r \tag{1.6}$$

can be used to determine the dependence of $\mathcal{G}$ on λ.

For this question the most interesting, because most singular, case is that of zero temperature, $T = 0$. For positive temperature or the finite-volume lattice case the expansion obtained by expanding the factor $e^{\lambda V}$ in λ is convergent, but its radius of convergence shrinks to zero in the thermodynamic and zero-temperature limit: at $T = 0$ and in infinite volume, one cannot even pose the question of convergence of the expansion in λ because the coefficients G_r already diverge for $r \geqslant 3$. In the limit $T \to 0$, (1.5) reduces to expectation values in the ground state of the system, so physically the question is about the nature of the many-particle ground state of the system and the validity of perturbation theory to calculate n-point functions. The radius of convergence of the unrenormalized expansion in finite volume shrinks to zero as the volume goes to infinity. Thus, although the expansion converges for the large but finite systems which these models are to describe, this is true only if λ is of order 1/volume, which is obviously unrealistic for any macroscopic system. Consequently, the unrenormalized expansion will not give insight into the properties of the ground state.

In this paper we consider formal perturbation theory. That is, we study the thermodynamic limit of the coefficient functions G_r. By an analysis similar to ref. 2, the expansion is renormalized so that these functions converge as the volume goes to infinity. More precisely, we introduce a well-defined infinite-volume model obtained by cutting off the singularity at the Fermi surface (i.e., introducing an infrared cutoff) and renormalized by including counterterms K in the action, and then show that all coefficients G_r have limits as the infrared cutoff is removed. Although we do not go through the finite-volume bounds here, it will be clear from the way our bounds are derived that the same procedure can be applied to obtain an expansion in finite volume with coefficients that converge in the thermodynamic limit. The counterterms are bilinear in the fermions and can therefore be viewed as a modification of H_0 (although they are treated as extra interaction vertices in the formal expansion). They also have finite limits as the infrared cutoff is removed. The addition of the counterterms K changes the free Hamiltonian H_0 to $\tilde{H}_0 = H_0 + K$, where K depends on λ and H_0. Thus the free part of the model is no longer fixed, but changes with λ as well: introducing counterterms changes the model. One can, however, obtain an infrared finite expansion for a prescribed free model, i.e., with $\tilde{H}_0$ prescribed, by solving $\tilde{H}_0 = H_0 + K(H_0)$ for H_0. This is far from straightforward because for the nonspherical Fermi surfaces that we study here, the counterterms are not just constants (such as a shift in the chemical potential μ), but functions, i.e., they change the one-particle kinetic energy operator T in a nontrivial way. Therefore, the equation for H_0 is an equation in a function space. In this paper, we prove that it has at most one solution H_0. In a separate paper,[10] we prove the existence of the solution. The equation relating H_0 and $\tilde{H}_0$ will also be discussed in more detail below.

Except for special cases, the renormalized expansion is, as an expansion in λ, not convergent but only locally Borel summable because the coefficients behave as $G_r \sim r!$. The occurrence of these factorials indicates that the nonperturbative ground state may exhibit symmetry breaking. For example, if the interaction is attractive in the zero angular momentum sector, this is the case.[3] One of the main results we shall prove here is that for a very wide class of models, and regardless of the sign of the interaction, the r factorials in individual graphs come only from ladder diagrams.

By "locally Borel summable" we mean here and in Theorem 1.2 that the Borel transform is analytic in a disk of strictly positive radius $R>0$. This does not imply that the function is "Borel summable" in the sense that it can be reconstructed uniquely from its Borel transform. For that, one would need, among other things, analyticity of the latter in a neighborhood of the entire positive real axis.

Renormalization has been done[2] for the continuum case where $T=-\Delta/2m$ and $U=0$. We shall refer to this case as the spherical case since the band structure (defined below) has an $O(d)$ rotational symmetry. The procedure for removing the divergences in the present case is similar to the spherical case in that we have to renormalize two-legged insertions. However, the present work is a nontrivial extension of ref. 2 because in contrast to the spherical case, the counterterms are not constants. In brief, subtracting functions is much more complicated than subtracting constants. In particular, the regularity properties of the counterterms are quite subtle.

In the remainder of this introductory section, we give a nonrigorous, physical discussion of why divergences occur and how they may be removed by renormalization. We hope that this will convince the reader, before going through all the details, that the renormalization subtractions are natural and the divergences of the naive expansion are artificial in these models. We state our main results in Section 1.5 and then discuss their physical interpretation. Finally, we give an overview of the sections containing the proofs. Every section begins with a brief explanation of what is done and how it fits into the general strategy.

1.2. The Formal Perturbation Expansion

The models have the formal functional integral representation

$$P(\eta, \bar{\eta}) = \int D\psi\, D\bar{\psi}\, e^{\mathscr{A} + (\bar{\eta}, \psi) + (\eta, \bar{\psi})} \tag{1.7}$$

where $\mathscr{A} = -(\bar{\psi}, C^{-1}\psi) - \lambda V$, $D\psi\, D\bar{\psi}$ is the formal measure $\prod_{x,\alpha} d\psi_\alpha(x)\, d\bar{\psi}_\alpha(x)$,

$$(\bar{\psi}, C^{-1}\psi) = \int ds(x)\, ds(y) \sum_{\alpha,\beta} \bar{\psi}_\alpha(x)(C^{-1})_{\alpha\beta}(x, y)\, \psi_\beta(y) \tag{1.8}$$

and

$$V = \int ds(x)\, ds(x') \sum_{\alpha,\beta,\alpha',\beta'} \bar{\psi}_\alpha(x)\, \psi_\beta(x)\, \tilde{v}_{\alpha\beta,\alpha',\beta'}(x, x')\, \bar{\psi}_{\alpha'}(x')\, \psi_{\beta'}(x') \tag{1.9}$$

where now $\int ds(x)\, F(x)$ stands for the integral over the spatial variable $\mathbf{x}$ and imaginary time τ, $x=(\tau, \mathbf{x})$, with an appropriate measure, e.g.,

$$ds(x) = d\tau\, d^d\mathbf{x} \tag{1.10}$$

for a continuous system on $[0, \beta] \times \mathbb{R}^d$ and

$$\int ds(x)\, F(x) = \int_0^\beta d\tau \sum_{\mathbf{x} \in \Gamma} F(\tau, \mathbf{x}) \tag{1.11}$$

for a lattice system on $[0, \beta] \times \Gamma$, e.g., $\Gamma = \mathbb{Z}^d$. Here $\beta = 1/k_B T$ is the inverse temperature. The imaginary time is introduced to get a functional integral representation for the trace over Fock space in the standard way. The connected Green functions can formally be calculated as derivatives of $\log P$ with respect to the sources η and $\bar\eta$.

In this paper, we consider the limiting case $T=0$, so $\beta=\infty$ and the configuration spaces are $\mathbb{R}^{d+1}$ and $\mathbb{R} \times \Gamma$ (e.g., $\mathbb{R} \times \mathbb{Z}^d$), respectively. The spin index is $\alpha \in \{\uparrow, \downarrow\}$, and the interaction is assumed to be translation invariant, so that

$$\tilde{v}_{\alpha\beta, \alpha'\beta'}(x, x') = v_{\alpha\beta, \alpha'\beta'}(\tau - \tau', \mathbf{x} - \mathbf{x}') \tag{1.12}$$

and is short range, i.e., v decreasing so fast that its Fourier transform $\hat{v}$ is at least twice differentiable (see Assumption A1 below). Note that we do not assume that it is instantaneous. For simplicity, we also assume that it is spin-diagonal, i.e., $v_{\alpha\beta, \alpha'\beta'} = \delta_{\alpha\alpha'} \delta_{\beta\beta'} v$. In contrast to the assumption about the decay of v, the latter assumption is merely for notational convenience and can easily be dropped.

One may imagine v to arise from exchange of (quasi)particles like photons or phonons and formalize this by a Hubbard–Stratonovich transformation, introducing one or more scalar fields with covariance v so that the interaction vertex is resolved as an exchange of fields and the interaction becomes bilinear in the fermion fields. For the purposes of the perturbation expansion we shall not need this. In particular, since we assume smoothness of $\hat{v}$, we shall not need a cutoff on the interaction lines, and we shall often draw graphs with four-legged vertices instead of ones with interaction lines.

For the lattice models, we take

$$(C^{-1})(x, x') = \delta_{\alpha\beta}(\delta_{\mathbf{x}\mathbf{x}'}(\partial_{\tau'} - \mu) - T_{\mathbf{x}-\mathbf{x}'})\, \delta(\tau - \tau') \tag{1.13}$$

where μ is the chemical potential and $T_{\mathbf{x}-\mathbf{x}'}$ is the amplitude for hopping from site $\mathbf{x}$ to site $\mathbf{x}'$, which we assume to the symmetric and short ranged (see Assumptions A2 and A3 on e below).

A model of particular interest that is easy to formulate but difficult to analyze is the Hubbard model, for which

$$T_{\mathbf{x}} = \sum_{|\mathbf{y}|=1} t_{\mathbf{y}} \delta_{\mathbf{x},\mathbf{y}} \tag{1.14}$$

with $t_{\mathbf{y}}$ the so-called hopping parameters. In the simplest version of the model, $t_{\mathbf{y}} = t$ is the same for all $\mathbf{y}$ of length one, so the operator T is just the discrete Laplacian on $\mathbb{Z}^d$, with the diagonal term omitted since it can be absorbed in the chemical potential μ, and the interaction term is one-site and spin-diagonal,

$$v_{\alpha\beta,\alpha'\beta'}(\mathbf{x}-\mathbf{x}') = \delta_{\mathbf{x}\mathbf{x}'} \delta_{\alpha\beta} \delta_{\alpha'\beta'} \tag{1.15}$$

Various extensions of this model, e.g., with more complicated finite-range hopping, have been studied in connection with high-temperature superconductivity. For suitable values of the filling factor, they all fall into the class of band structures discussed here. For a review of mathematically rigorous results about the Hubbard model, see ref. 4.

Formally equivalent to P, but in fact much more convenient is the generating functional for connected amputated Green functions

$$\mathscr{G}(\psi_e, \bar{\psi}_e) = \log \frac{1}{Z} \int D\psi \, D\,\bar{\psi}\, e^{-(\bar{\psi}, C^{-1}\psi)} e^{\lambda V(\psi+\psi_e, \bar{\psi}+\bar{\psi}_e)} \tag{1.16}$$

where the constant Z takes out the field-independent term so that $\mathscr{G}(0,0) = 0$. As written above, $\mathscr{G}$ is not a well-defined object in infinite volume; it can be made well defined by restricting to a finite volume Λ or by introducing a suitable cutoff. If the free covariance C is bounded and any power of it is integrable, $(1/|\Lambda|)\,\mathscr{G}_\Lambda$ exists and is analytic in λ, as was first observed by Caianiello. However, for any realistic model, C will not have these properties unless cutoffs are imposed. The radius of convergence obtained using naive bounds shrinks to zero when the cutoffs are removed, and establishing analyticity uniformly in the cutoffs requires techniques as in ref. 5.

Our analysis is done in momentum space, where from now on momentum is short for Bloch's quasimomentum, which can be used to label one-particle states because of the periodicity of the one-article potential U. In infinite volume, momentum space is the first Brillouin zone $\mathscr{B}$, i.e., the torus

$$\mathscr{B} = \mathbb{R}^d / \Gamma^{\#} \tag{1.17}$$

where $\Gamma^{\#}$ is the dual lattice to Γ, e.g., $\Gamma^{\#} = 2\pi\mathbb{Z}^d$ for $\Gamma = \mathbb{Z}^d$. In finite volume, the momenta are in a finite subset of $\mathscr{B}$, $\mathbf{p} = 2\pi\mathbf{n}/L$ with $\mathbf{n} \in \mathbb{Z}^d \cap \mathscr{B}$ if the volume is a box of side length L. The eigenfunction expansions used to transform into momentum space are discussed briefly in Appendix B for the general case; for the purposes of this introduction, we just give the formulas for the case of a lattice model on $\mathbb{Z}^d$, where we can simply do a Fourier expansion. The only changes in the general case are (of course) that the Brillouin zone will differ with the lattice and that the formulas for switching between position and quasimomentum space involve the eigenfunctions of the one-particle Hamiltonian H_0 with the periodic potential. Under the Fourier transform

$$\psi(x) = (2\pi)^{-(d+1)} \int d^d\mathbf{p}\, dp_0\, e^{-ip_0\tau + i\mathbf{px}}\, \hat{\psi}(p)$$
$$\bar{\psi}(x) = (2\pi)^{-(d+1)} \int d^d\mathbf{p}\, dp_0\, e^{ip_0\tau - i\mathbf{px}}\, \hat{\bar{\psi}}(p) \tag{1.18}$$

the quadratic part of the action becomes

$$(\bar{\psi}, C^{-1}\psi) = (2\pi)^{-(d+1)} \int d^d\mathbf{p}\, dp_0\, \bar{\psi}(p)(ip_0 - e(\mathbf{p}))\, \psi(p) \tag{1.19}$$

where we have dropped the carets and introduced the band structure

$$e(\mathbf{p}) = \varepsilon(\mathbf{p}) - \mu \tag{1.20}$$

where

$$\varepsilon(\mathbf{p}) = \int ds(\mathbf{x})\, e^{-i\mathbf{px}} T_{\mathbf{x}} \tag{1.21}$$

and, with $p_i = ((p_i)_0, \mathbf{p}_i)$ and

$$d^{d+1}p = dp_0\, d^d\mathbf{p} = \frac{dp_0}{2\pi} \frac{d^d\mathbf{p}}{(2\pi)^d} \tag{1.22}$$

the interaction becomes

$$V = \int d^{d+1}p_1 \dots d^{d+1}p_4 (2\pi)^{d+1} \delta((p_2 + p_4 - p_1 - p_3)_0)$$
$$\times \delta^{\#}(\mathbf{p}_2 - \mathbf{p}_1 + \mathbf{p}_4 - \mathbf{p}_3)$$
$$\times \hat{v}(\mathbf{p}_3 - \mathbf{p}_1)\, \bar{\psi}(p_1)\, \psi(p_2)\, \bar{\psi}(p_3)\, \psi(p_4) \tag{1.23}$$

Here $\delta^{\#}$ is the delta function on $\mathscr{B}$; more explicitly,

$$\delta^{\#}(\mathbf{p}) = \frac{1}{(2\pi)^d} \sum_{\mathbf{x} \in \mathbb{Z}^d} e^{i\mathbf{px}} = \sum_{\gamma \in \Gamma^{\#}} \delta(\mathbf{p} + \gamma) \tag{1.24}$$

where the δ on the right side denotes that on $\mathbb{R}^d$. In general, the solution of the one-particle problem will produce crossing bands. We exclude this case here, and we also introduce an ultraviolet cutoff that removes the high-energy bands. For the lattice systems, such a cutoff is already built in as the lattice spacing; for continuous systems, it is not a real physical restriction since high energies do not occur in a crystal. If there are finitely many bands that do not cross, the band index is just a bookkeeping device dragged along, so, without loss, we restrict to the one-band case here.

For $\lambda = 0$, the fermions do not influence each other and the model is completely characterized by the covariance C,

$$\check{C}(x) = \int d^{d+1}p \, \frac{e^{-ip_0\tau + i\mathbf{p}\mathbf{x}}}{ip_0 - e(\mathbf{p})} \tag{1.25}$$

in the sense that all $2n$-point functions are determinants of matrices with elements $\check{C}(x_i - x_j)$.

The propagator in momentum space, $C(p) = e^{ip_0 0^+}/(ip_0 - e(\mathbf{p}))$, has a singularity at $p_0 = 0$ for all $\mathbf{p} \in S$, where $S = \{\mathbf{p}: e(\mathbf{p}) = 0\}$ is the Fermi surface of the independent-electron approximation. Although the function $1/(ip_0 - e(\mathbf{p}))$ is in $L^{1+\delta}_{\text{loc}}(\mathbb{R} \times \mathscr{B})$ for all $\delta \in [0, 1)$, graphs in the perturbation expansion diverge because of the singularity on S and because in the expansion, arbitrary powers of C are integrated. The numerator $e^{ip_0 0^+}$ is included in the standard way since we want to consider the expansion around the situation where all states inside the Fermi surface, i.e., those with $e(\mathbf{p}) < 0$, are already occupied.

Expanding $\mathscr{G}$ in a formal power series in λ, we can write

$$\mathscr{G}(\psi_e, \bar{\psi}_e) = \sum_{r \geqslant 0} \lambda^r \mathscr{G}_r(\psi_e, \bar{\psi}_e) \tag{1.26}$$

with

$$\mathscr{G}_r(\psi, \bar{\psi}) = \sum_{m \geqslant 1} \sum_{\substack{\alpha_1, \ldots, \alpha_m \\ \bar{\alpha}_1, \ldots, \bar{\alpha}_m}} \int \prod_{i=1}^{2m} d^{d+1}p_i (2\pi)^{d+1} \delta^{\#}\left(\sum_{i=1}^{m} p_i - \sum_{i=m+1}^{2m} p_i \right) \times (G_{2m,r})_{\alpha_1, \ldots, \bar{\alpha}_m}(p_1, \ldots, p_{2m-1}) \prod_{i=1}^{m} \psi_{\alpha_i}(p_i) \, \bar{\psi}_{\bar{\alpha}_i}(p_{m+i}) \tag{1.27}$$

where the coefficient function $G_{2m,r}$ is totally antisymmetric in the simultaneous exchange of momenta and spin indices (see Section 2.3). Again, the $\delta^{\#}$ is periodic with respect to $\Gamma^{\#}$ in the spatial part of the momentum. The coefficient $G_{2m,r}$ can be expressed in the usual way as a sum over values of connected Feynman diagrams. The sum over m runs over a finite index set for each fixed r because the number of vertices is r and the graphs are connected with $2m$ external legs.

The Feynman graphs are similar to those in quantum electrodynamics: there are two types of lines, namely fermion lines (drawn solid), carrying a direction, and interaction lines (drawn dashed). The vertices have two legs to which fermion lines can be connected (one incoming, one outgoing) and one leg for an interaction line. The action determines the assignment of propagators $C(p)$ to fermion lines, $\hat{v}(p)$ to interaction lines, and momentum conservation delta functions to vertices. Equivalently, one can replace two vertices that are joined by an interaction line by a single four-fermion vertex with exactly two incoming fermion legs and exactly two outgoing fermion legs. The graphs then have only four-legged fermion vertices and only fermion lines. There is one notable difference between the cases $U=0$ and $U\neq 0$: In the spherical case ($U=0$), where $\varepsilon(\mathbf{p})=\mathbf{p}^2/2m$, $\mathbf{p}\in\mathbb{R}^d$. The corresponding ultraviolet problem (behavior at large $|\mathbf{p}|$) was solved in ref. 2. In the presence of a crystal potential ($U\neq 0$), the integrals over the spatial part of the momentum are over the first Brillouin zone $\mathscr{B}$, which is a compact set. Thus there is no case of large $\mathbf{p}$ here. Momentum conservation at every vertex means conservation in $\mathscr{B}$, as given by $\delta^{\#}$ above. If one prefers to think of the momenta in $\mathbb{R}^d$, fixing momenta with $\delta^{\#}$ means that at every vertex, there remains a sum over $\gamma\in\Gamma^{\#}$. Although formally infinite, this sum always contains only one nonzero term since there is a unique $\gamma\in\Gamma^{\#}$ that translates back a vector in $\mathbb{R}^d$ into the fundamental domain of the translational group $\Gamma^{\#}$. However, it is natural and simpler to consider momentum space as the torus $\mathscr{B}$ since e is $\Gamma^{\#}$-periodic.

For example, in the Hubbard model,

$$e(\mathbf{p})=2t\sum_{i=1}^{d}\cos p_i-\mu \tag{1.28}$$

is the tight-binding band relation and $\hat{v}(\mathbf{p})=1$.

The much more general class of models and the range of chemical potential μ that we treat in this paper are given by the following assumptions.

1.3. Assumptions

We assume that the one-particle problem (discussed in Appendix B) is such that we have a Brillouin zone $\mathscr{B}$ which is a d-dimensional torus of type (1.17). We assume that $e = \varepsilon - \mu$ [see (1.20)] is a continuous function on $\mathscr{B}$ and that for some value μ_0 of the chemical potential, the Fermi surface

$$S = \{\mathbf{p} \in \mathscr{B}\colon e(\mathbf{p}) = 0\} \tag{1.29}$$

has only a finite number of connected components. Furthermore, there are $k \geqslant 2$ and a neighborhood $\mathscr{N}$ of S such that:

A1 The interaction $\hat{v} \in C^k(\mathbb{R} \times \mathscr{B}, \mathbb{C})$. The sup norm over $\mathbb{R} \times \mathscr{B}$ of the first k derivatives is finite. $\hat{v}$ has the symmetry $\hat{v}(-p_0, \mathbf{p}) = \hat{v}(p_0, \mathbf{p})$.

A2 The band structure $e \in C^k(\mathscr{N}, \mathbb{R})$, and $\nabla e(\mathbf{p}) \neq 0$ for all $\mathbf{p} \in S$.

The third assumption is a geometrical condition on the Fermi surface. It is very simple to understand and is fulfilled for generic surfaces. Let $n\colon S \to \mathbb{R}^d$, $\omega \mapsto n(\omega) = (\nabla e/|\nabla e|)(\omega)$, be the unit normal to the surface. By A2, S is a C^k submanifold of $\mathscr{B}$, and n is a C^{k-1} unit vector field. If S consists of more than one connected component, choose a normal field for any component. For $\omega, \omega' \in S$, define the angle between $n(\omega)$ and $n(\omega')$ by

$$\theta(\omega, \omega') = \arccos(n(\omega) \cdot n(\omega')) \tag{1.30}$$

Let

$$\mathscr{D}(\omega) = \{\omega' \in S\colon |n(\omega) \cdot n(\omega')| = 1\} = \{\omega' \in S\colon n(\omega) = \pm n(\omega')\} \tag{1.31}$$

and denote the $(d-1)$-dimensional measure of $A \subset S$ by $vol_{d-1} A$. Also, for any $A \subset \mathbb{R}^d$ and $\beta > 0$, denote by $U_\beta(A) = \{\mathbf{p} \in \mathbb{R}^d\colon \text{distance}(\mathbf{p}, A) < \beta\}$ the open β-neighborhood of A. For fixed ε and μ_0, we assume:

A3 There is an open interval $\mathscr{M}$ around μ_0 and there are strictly positive numbers Z_0, Z_1, ρ, β_0, and κ such that for all $\mu \in \mathscr{M}$, the Fermi surface $S = S(\mu) = \{\mathbf{p} \in \mathscr{B}\colon\ e(\mathbf{p}) = 0\}$ has the following properties: $S(\mu) \subset \mathscr{N}$, and for all $\beta \leqslant \beta_0$ and all $\omega \in S$:

(i) $vol_{d-1}(U_\beta(\mathscr{D}(\omega)) \cap S) \leqslant Z_0 \beta^\kappa$.

(ii) If $\omega' \notin U_\beta(\mathscr{D}(\omega)) \cap S$, then

$$|\sin \theta(\omega, \omega')| = [1 - (n(\omega) \cdot n(\omega'))^2]^{1/2} \geqslant Z_1 \beta^\rho$$

Throughout this paper, A1–A3 will be assumed to hold, and μ will be assumed to lie in the interval $\mathscr{M}$ specified in A3. We now explain what these assumptions mean.

Assumption A1 on $\hat{v}$ is a decay assumption in position space, e.g., for an instantaneous interaction V on a lattice system on $\mathbb{Z}^d$ and $k=2$, A1 holds if

$$\sum_{\mathbf{x} \in \mathbb{Z}^d} |\mathbf{x}|^2 \, |V(\mathbf{x})| < \infty \tag{1.32}$$

For continuous systems, A1 is implied by a similar integral condition.

Assumption A2 excludes singular points. For example, a point $\mathbf{p}$ on S where $\nabla e(\mathbf{p}) = 0$ is called a van Hove singularity.

The condition that e is continuously differentiable is fulfilled for the case where e comes from a Schrödinger equation for the one-body problem with a regular periodic potential, if there is no level crossing. Indeed, it is real analytic. In lattice models with finite-range hopping, e is analytic. However, infinite-range hopping is also allowed: $e \in C^k$ if the kth moment of the hopping amplitude exists, i.e., $\sum_{\mathbf{x}} |\mathbf{x}|^k \, |T_{\mathbf{x}}| < \infty$.

Assumption A3 is, more informally, that for every $\omega \in S$:

(i) The set of points ω' where the normal $n(\omega')$ is parallel or antiparallel to $n(\omega)$ has positive codimension $\kappa > 0$ in S.

(ii) If ω' is not in the set $\mathscr{D}(\omega)$, where the normal is (anti)parallel to $n(\omega)$, the angle between $n(\omega)$ and $n(\omega')$ increases with some power of the distance between ω' and $\mathscr{D}(\omega)$.

Thus in order to violate these assumptions, the surface S must have flat regions or subsets where $\theta(\omega, \omega')$ vanishes exponentially fast as $|\omega - \omega'| \to 0$. To illustrate A3, we show in Fig. 1 an example of a Fermi surface in $d=2$ (i.e., a Fermi curve) on $\mathscr{B} = \mathbb{R}^2/2\pi\mathbb{Z}^2$ that satisfies A3. In Fig. 1, the square

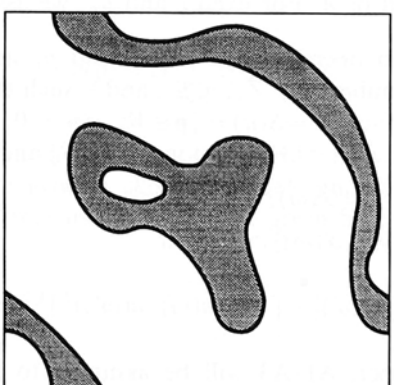

Fig. 1. An example of a Fermi surface obeying Assumption A3.

bounds the fundamental region $[-\pi, \pi)^2$ for the torus $\mathscr{B}$, and the shaded areas indicate $e(\mathbf{p}) < 0$.

Assumptions A2 and A3 imply the following bound, which we shall use in the proofs.

Volume Improvement Estimate. There is $\varepsilon > 0$ and there is a constant C_{vol} such that for all $\mu \in \mathscr{M}$ and for all $\eta_1 > 0$, $\eta_2 > 0$, $\eta_3 > 0$

$$I_2(\eta_1, \eta_2, \eta_3) \leqslant C_{\text{vol}}\, \eta_1 \eta_2 \eta_3^{\varepsilon} \tag{1.33}$$

where

$$\begin{aligned} I_2(\eta_1, \eta_2, \eta_3) = \sup_{\mathbf{q} \in \mathscr{B}} \max_{v_1, v_2 \in \{1, -1\}} & \int_{\mathscr{B} \times \mathscr{B}} d^d\mathbf{p}_1\, d^d\mathbf{p}_2 \\ & \times 1(|e(\mathbf{p}_1)| < \eta_1)\, 1(|e(\mathbf{p}_2)| < \eta_2) \\ & \times 1(|e(v_1 \mathbf{p}_1 + v_2 \mathbf{p}_2 + \mathbf{q})| < \eta_3) \end{aligned} \tag{1.34}$$

Here $1(E)$ denotes the indicator function of the event E, i.e., $1(E) = 1$ if E is true and $1(E) = 0$ otherwise. The additional factor η_3^{ε} will be called the volume improvement factor. The function I_2 allows us to give sharp bounds for arbitrary graphs based on a simple characterization of graphs (explained below).

Proposition 1.1. Assumptions A3 and A2 imply (1.33), with

$$\varepsilon \geqslant \frac{\kappa}{\kappa + \rho} \tag{1.35}$$

Proof. See Appendix A. ∎

Assumption A3, and thus (1.33), hold in particular if the set of filled states $\{\mathbf{p}: e(\mathbf{p}) \leqslant \mu\}$ is strictly convex and nowhere exponentially flat in the sense mentioned above. Thus the class of models with $\varepsilon > 0$ contains all those where the band structure is a strictly convex analytic function or a strictly concave analytic function, because, by definition, the sets $\{\mathbf{p}: e(\mathbf{p}) \leqslant \mu\}$ are then strictly convex sets, and the Fermi surface is just the boundary of such a set. By analyticity, exponential or complete flatness is excluded in this case. This is obviously a very natural condition since essentially all band structures of practical importance in solid-state models are strictly convex around the band minimum, and so our results apply to the case where the Fermi edge is just above the minimum of a band.

The proof we give in the Appendix also shows that this nonnestedness is essentially a transversality condition on the Fermi surface and its translates—hence the need to have some control over the set $\mathscr{D}(\omega)$, which is essentially the set where the intersection would not be transversal.

Examples. 1. The spherical band structure $e(\mathbf{p}) = \mathbf{p}^2/2m - \mu$ fulfills all these hypotheses for any $\mu > 0$, with $\rho = 1$ and $\kappa = d-1$.

2. The Hubbard model with tight-binding band structure (1.28) fulfills A1–A3 for all $\mu \neq 0$, i.e., away from half-filling, with $\kappa = d-1$. If the band is either empty or full (cases which are of little physical interest), the volume shrinks even faster in $d \leqslant 2$. For the half-filled case $\mu = 0$, both A2 and A3 fail. Assumption A2 is not fulfilled because of the van Hove singularities at the boundary of $[-\pi, \pi]^d$, and A3 does not hold because the surface has flat regions (in $d = 2$ it is diamond-shaped). This is an example where a nongeneric (because flat) surface plays a role in a physical model.

It is well known that the half-filled band is a very special case, and that this is due to the nesting we just discussed, as well as to the presence of van Hove singularities. A physical way of understanding this is that the particle–hole symmetry restricts the shape of the Fermi surface. More generally speaking, van Hove singularities must always occur at some values of μ for topological reasons: for generic e, the condition $\nabla e(\mathbf{p}) = 0$ is satisfied at isolated points $\mathbf{p} \in \mathscr{B}$. Thus there is a van Hove singularity for each value of μ for which the corresponding Fermi surface passes through one of these points. By way of contrast, nesting in the sense that A3 fails is a much more restrictive condition on $e(\mathbf{p})$. Stated differently, a nesting condition requires fine tuning of e. The occurrence of flat parts of S and van Hove singularities at the same value of μ ($\mu = 0$, half-filling) in the Hubbard model with the band structure (1.28) is accidential. They no longer occur at the same value of μ if next-to-nearest neighbor hopping is allowed.

3. For $d = 2$, $S = \{(x, y) \colon x^{2n} + y^{2n} = 1\}$ is another example. Here $\rho = 2n - 1$ and, as in Examples 1 and 2, $\kappa = d - 1 = 1$. As $n \to \infty$, S approaches $\{(x, y) \colon |x| = 1 \text{ or } |y| = 1, (x^2 + y^2)^{1/2} \leqslant \sqrt{2}\}$, which is flat away from its edges, and the lower bound for the volume improvement exponent ε goes to zero like $1/n$ by (1.35).

4. The two-torus imbedded in $\mathbb{R}^3$ is an example with $\rho = 1$ and $\kappa = 1$. The codimension κ is only 1 in this example because $\mathscr{D}(\omega)$ may be a union of two circles for some ω.

5. The surface $e^{-1/x^2} + e^{-1/y^2} = e^{-1}$ is an example where, due to the essential singularity at (0, 1), the condition A3 does not hold. As discussed above, under some regularity conditions on the one-particle problem, such surfaces are ruled out. We may well expect that they will not occur in any realistic model.

1.4. Divergences and Hartree–Fock Theory

Under the assumptions stated above, the only source of divergences in perturbation theory is exactly the same as in the spherical case, where e is given by $e(\mathbf{p}) = \mathbf{p}^2/2m - \mu$, namely the accumulation of powers of the propagator due to strings of two-legged subdiagrams: the function $C(p) = (ip_0 - e(\mathbf{p}))^{-1}$ is singular on the set $\{0\} \times S$. By the assumption that ∇e does not vanish on S and by compactness of S, we can introduce coordinates $\rho = e(\mathbf{p})$ and ω, where ω parametrizes the submanifold $S_\rho = \{\mathbf{p}: e(\mathbf{p}) = \rho\}$ (this works in a neighborhood of $S = S_0$, i.e., for $|\rho| \leqslant \rho_0$). Thus $\mathbf{p} = \phi(\rho, \omega)$ in this neighborhood, and for $\alpha < 2$,

$$\begin{aligned}\int_{|ip_0 - e(\mathbf{p})| < \rho_0} dp_0\, d^d\mathbf{p}\, \frac{1}{|ip_0 - e(\mathbf{p})|^\alpha} &= \int_{|ip_0 - \rho| < \rho_0} dp_0 d\rho\, d^{d-1}\omega\, \frac{1}{|ip_0 - \rho|^\alpha}\, |\det \phi'(\rho, \omega)| \\ &= \int_0^{\rho_0} \frac{r\, dr}{r^\alpha} F(r) \qquad (1.36)\end{aligned}$$

where

$$F(r) = \int_0^{2\pi} d\theta \int d^{d-1}\omega\, |\det \phi'(r\cos\theta, \omega)|$$

The integral converges for $\alpha < 2$, but since $F(r) \geqslant f_0 > 0$ for all r, it diverges for $\alpha \geqslant 2$.

By the Feynman rules, graphs like the ones shown in Fig. 2 diverge, because, e.g., the value of the first one would be $\int dp_0 \int d^d\mathbf{p}\, C(p)^3 T(p)^2$ $\hat{v}(q - p)$ for external momentum q, which diverges because the third power of the propagator appears, so $\alpha = 3$, and

$$T(0, \mathbf{q}) = \int d^{d+1}p \hat{v}((0, \mathbf{q}) - p)\, C(p)$$

will not vanish on the Fermi surface S where the propagator is singular. The proof that these two-legged insertions are the only source of divergences of values of individual graphs was given in ref. 2 for the spherical case, and a similar result holds in the present case (see Section 2.1). The only way a divergence could be absent is that the function $T(p)$ also vanishes on the Fermi surface. However, this will not happen by itself in general. Renormalization is done by subtracting $(\ell T)(p) = T(0, \mathbf{P}(\mathbf{p}))$ for

Fig. 2. Graphs with two-legged insertions.

any two-legged insertion $T(p)$, where $\mathbf{P}(\mathbf{p})$ is the projection of the vector $\mathbf{p}$ onto the Fermi surface S, for $\mathbf{p}$ in a fixed neighborhood of S. The precise definition of $\mathbf{P}$ is given in Section 2.2; it is defined by taking a vector field u that is transversal to S in the sense that $|(u \cdot \nabla e)(\mathbf{p})| \geqslant u_0 > 0$ for all $\mathbf{p} \in S$, and taking $\mathbf{P}(\mathbf{p})$ to be the point where the integral curve of u through $\mathbf{p}$ intersects S (see Fig. 3). The reader familiar with resummation methods based on the Schwinger–Dyson equations in solid-state theory, e.g., the Hartree–Fock method, may ask why one never sees these divergences in the integral equations corresponding to these approximations, although they are said to resum part of those diagrams which appear to be ill defined in the formal perturbation expansion. This point actually gives a hint at what renormalization in these models does. Consider the Hartree–Fock approximation,[6] as given by the integral equation for the two-point function

$$(G_2)_{\alpha\alpha'}(\tau_1, \mathbf{x}_1, \tau_2, \mathbf{x}_2) = \langle \psi_\alpha(\tau_1, \mathbf{x}_1)\, \bar{\psi}_{\alpha'}(\tau_2, \mathbf{x}_2) \rangle \tag{1.37}$$

which reads, denoting $x = (\tau, \mathbf{x})$,

$$\begin{aligned} G_2(x_1, x_2) = \check{C}(x_1, x_2) &+ \lambda \int ds(x)\, ds(x') \\ &\times \check{C}(x_1, x)\, v(x' - x)\, G_2(x, x')\, G_2(x', x_2) \\ &- \lambda \int ds(x)\, ds(x')\, \check{C}(x_1, x)\, G_2(x, x_2)\, v(x' - x) \operatorname{tr} G_2(x', x') \end{aligned} \tag{1.38}$$

where the trace is over spin indices. Representing the free propagator $\check{C}$ by a thin solid line, interaction lines by dashed lines, and the interacting propagator G_2 by a thick line, this equation can be depicted as in Fig. 4, from which it is evident that by iterating the equation one produces a resummation to all orders that includes graphs without polarization effects, in particular some that are divergent in the formal perturbative expansion,

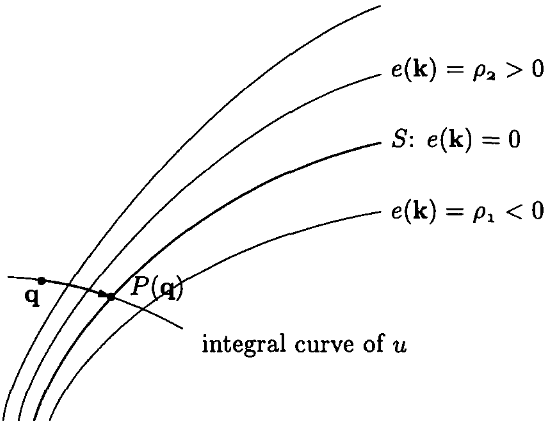

Fig. 3. The projection to the Fermi surface.

as for instance the first one in Fig. 2. However, the whole point of the "resummation" is to avoid summation, instead making the ansatz

$$\hat{G}_2(p) = e^{ip_0 0^+}(ip_0 - e(\mathbf{p}) - \Sigma(p))^{-1} \tag{1.39}$$

and rewriting the integral equation in momentum space (in the translation-invariant case) as

$$\begin{aligned}\Sigma(q) = & -\lambda \int d^{d+1}p \, \frac{\hat{v}(q-p)\, e^{ip_0 0^+}}{ip_0 - e(\mathbf{p}) - \Sigma(p)} \\ & + \lambda \hat{v}(0) \operatorname{tr} \int d^{d+1}p \, \frac{e^{ip_0 0^+}}{ip_0 - e(\mathbf{p}) - \Sigma(p)}\end{aligned} \tag{1.40}$$

For a reasonable function Σ, the singularity of $\hat{G}_2$ is again integrable by the argument of (1.36). However, $\hat{G}_2$ will be singular if $p_0 = 0$ and $e(\mathbf{p}) + \Sigma(0, \mathbf{p}) = 0$, rather than if $p_0 = 0$, $e(\mathbf{p}) = 0$, so S is not the Fermi surface of the interacting system. If one attempted to seek the solution of (1.40) by an expansion of Σ in powers of λ and exchanged summation and

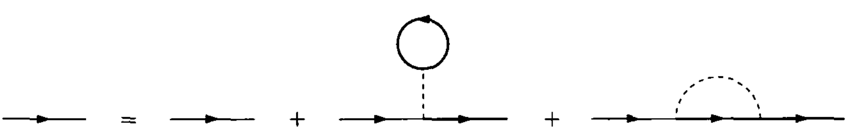

Fig. 4. The Hartree–Fock equations.

integrals, one would run into divergent expressions for well-defined integrals, such as

$$\int d^{d+1}p\, \hat{G}_2(p) \text{ “=” } \sum_{n=0}^{\infty} \int d^{d+1}p \frac{\Sigma(p)^n}{(ip_0 - e(\mathbf{p}))^{n+1}} \tag{1.41}$$

These divergences are part of those of the unrenormalized perturbation expansion.

The conclusion of this discussion is that in fact not the subtractions but the divergences are artificial, because they come from expanding a moving singularity in terms of a fixed one, and that the counterterms that are added to the action to implement renormalization have something to do with $\Sigma(0, \mathbf{p})$. This is the main idea; it remains to be shown that this is really so for the exact theory, where there is no such simple integral equation for the self-energy Σ as (1.40) for the Hartree–Fock approximation. For instance, the Hartree–Fock resummation does not include any polarization effects and thus differs from the exact result already in second order. It is necessary to include those effects for renormalization, e.g., the second graph in Fig. 2 also contributes a formally divergent term to $\mathscr{G}_2$ and thus needs to the renormalized. The Hartree–Fock graphs will, however, turn out to be spcial in that they are the only two-legged graphs that are nonoverlapping to all scales. Also, the graphs contributing to the (one-particle-irreducible) Hartree–Fock self-energy have the property that the external momentum can always be routed through an interaction line. Thus the degree of differentiability of the Hartree–Fock approximation to Σ, as defined by (1.40), with respect to the external momentum is the same as that of the interaction $\hat{v}$. For the exact self-energy Σ, the answer is not so easy.

In the four-legged case, the nonoverlapping graphs, i.e., those without improved power counting, will turn out to be the ladder graphs that are known to produce symmetry breaking.[3,5] We will give an explicit bound that shows that only insertions of these four-legged diagrams can produce the factorials in the values of individual graphs. The concept of improved power counting, together with this result, makes precise the notion of leading divergences (see Section 2.7).

The subtractions are implemented by adding counterterms to the action. These counterterms are of mass type, that is, they are bilinear in the fermion fields. If both the band structure and the potential have spherical symmetry, any two-legged diagram contributes a value $T(p_0, \mathbf{p}) = T(p_0, |\mathbf{p}|)$ to the two-point function, i.e., spherical symmetry forces the function only to depend on $|\mathbf{p}|$. Thus the subtracted terms are simply constants since

$$T(0, \mathbf{P}(\mathbf{p})) = T(0, |\mathbf{P}(\mathbf{p})|) = T(0, \sqrt{2m\mu}) \tag{1.42}$$

for the spherical band structure $e(\mathbf{p}) = \mathbf{p}^2/2m - \mu$. Their sum produces a shift in the chemical potential[2] and the interpretation of renormalization in that case is that the interaction changes the radius of the Fermi sphere. In the case where the original band structure or the potential does not have spherical symmetry, $T(0, \mathbf{P}(\mathbf{p}))$ is still a function of the spatial part of momentum. This is easy to understand since in that case the shape of the Fermi surface may change, but technically it is a complication because, in renormalization group language, there is not only one relevant parameter, but instead there are infinitely many, needed to describe the shape of the surface. To cancel the divergences, the counterterms are chosen such that the interacting Fermi surface is held fixed. They determine the shift between the noninteracting and the interacting Fermi surface and thus include part of the effects of the self-energy.

1.5. Results

The long-distance behavior of the free electron Green function $\check{C}(x-y)$ is a power-law falloff in $|x-y|$, determined by the singularity of $C(p)$ in momentum space. If one cuts off this singularity, i.e., forbids small values of the energy $e(\mathbf{p})$, the Green function decays exponentially, with a decay length ~ 1/energy. We do a multiscale analysis by decomposing into energy shells and successively integrating out fields in those energy shells. This gives rise to a series of effective actions, which can also be viewed as the Green functions with an infrared cutoff given by the energy scale. Let $M > 1$ be a scale parameter (see Section 2.1), and $j \in \mathbb{Z}$, $j < 0$. The shell of scale j around the Fermi surface is the set of p for which $M^{j-2} \leqslant |ip_0 - e(\mathbf{p})| \leqslant M^j$. We consider an infrared cutoff on scale M^I, where $I > -\infty$, $I \in \mathbb{Z}$, $I < 0$ (see Section 2.1). We also call I the infrared cutoff. Let $\mathcal{M}$ be the interval given in Assumption A3 and fix $\mu \in \mathcal{M}$. Define the connected amputated renormalized $2m$-point Green functions G^I_{2m} with infared cutoff I as the formal power series

$$G^I_{2m} = \sum_{\lambda=1}^{\infty} \lambda^r G^I_{2m,r} \tag{1.43}$$

where $G^I_{2m,r}$ is the renormalized rth-order Green function [see (2.72)]. Without going into the details, it is the modification of the connected Green function in (1.27) where only the fields with energy scale $\geqslant M^I$ are integrated over, and the interaction contains an additional term K^I that modifies the band structure. The term "renormalized" refers to this modification since in the graphical analysis, K^I appears as the counterterms. We also introduce an ultraviolet cutoff that removes the higher bands. The G^I_{2m} are analytic in λ for $I > -\infty$, and we will show that the limit $I \to -\infty$, $G_{2m,r}$, of $G^I_{2m,r}$ exists, so that in that limit (1.43) becomes a well-defined *formal* power series. For $s \geqslant 0$ and functions $F\colon \mathbb{R} \times \mathscr{B} \times \{\uparrow, \downarrow\} \to \mathbb{C}$ define the norms

$$|F|_s = \sum_{\alpha: |\alpha| \leqslant s} \sup_{p \in \mathbb{R} \times \mathscr{B}} \max_{\sigma, \sigma' \in \{\uparrow, \downarrow\}} |\partial^\alpha F_{\sigma\sigma'}(p)| \tag{1.44}$$

where $\alpha = (\alpha_0, ..., \alpha_d) \in \mathbb{Z}^{d+1}$ is a multiindex with $\alpha_i \geqslant 0$ for all i, $|\alpha| = \sum_{i=0}^d \alpha_i$, and

$$\partial^\alpha = \left(\frac{\partial}{\partial p_0}\right)^{\alpha_0} \cdots \left(\frac{\partial}{\partial p_d}\right)^{\alpha_d}$$

Similarly, for functions u defined on $(\mathbb{R} \times \mathscr{B})^{n-1} \times \{\uparrow, \downarrow\}^n$, define

$$|u|_s = \sup \left\{ \sum_{\substack{\alpha_1, ..., \alpha_{n-1} \\ |\alpha_1| + \cdots + |\alpha_{n-1}| \leqslant s}} |D^\alpha u_A(p_1, ..., p_{n-1})| \colon p_i \in \mathbb{R} \times \mathscr{B},\ A \in \{\uparrow, \downarrow\}^n \right\} \tag{1.45}$$

where

$$D^\alpha = \left(\frac{\partial}{\partial p_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial p_{n-1}}\right)^{\alpha_{n-1}}$$

and

$$|u|' = \int_{(\mathbb{R} \times \mathscr{B})^{n-1}} d^{d+1}p_1 \ldots d^{d+1}p_{n-1} \max_{A \in \{\uparrow, \downarrow\}^n} |u_A(p_1, ..., p_{n-1})| \tag{1.46}$$

The self-energy $\Sigma^I = \sum_{r \geqslant 1} \lambda^r \Sigma^I_r$ is given as a formal power series by

$$\Sigma^I(\mathbf{p}) = (1 - G^I_2 C_I)^{-1} G^I_2(\mathbf{p}) \tag{1.47}$$

where C_I is the propagator with infrared cutoff I [the inverse relation is $G^I_2 = \Sigma^I(1 - C_I \Sigma^I)^{-1}$].

Theorem 1.2. There is a formal power series

$$K^I(\mathbf{p}) = \sum_{r=1}^{\infty} K^I_r(\mathbf{p})\, \lambda^r \tag{1.48}$$

such that for the interaction

$$\mathscr{V} = \lambda V + \int d^{d+1}p\, \bar{\psi}(p)\, K^I(\mathbf{p})\, \psi(p) \tag{1.49}$$

the following statements hold. For all $m \in \mathbb{N}$, the infrared limit $I \to -\infty$ of the $G^I_{2m,r}$ exists. More precisely, for every $r \geqslant 1$, there are $\Sigma_r \in C^1(\mathbb{R} \times \mathscr{B}, \mathbb{C})$ and $K_r \in C^1(\mathscr{B}, \mathbb{R})$ and for all $m \geqslant 1$, there are $G_{2m,r}$ such that, as $I \to -\infty$:

(i) $G^I_{2,r} \to G_{2,r}$ in $|\cdot|_0$.

(ii) $G^I_{2m,r} \to G_{2m,r}$ converges in $|\cdot|'$.

(iii) $\Sigma^I_r \to \Sigma_r$ in $|\cdot|_1$ and $(\ell\Sigma)(p) = \Sigma(0, \mathbf{P}(\mathbf{p})) = 0$.

(iv) $K^I_r \to K_r$ in $|\cdot|_1$.

Moreover, the Green functions are locally Borel summable, that is, there are constants Γ_2, κ_2, σ_2, and Γ_{2m} such that

$$\begin{aligned} |G_{2,r}|_0 &\leqslant \Gamma_2^r \cdot r! \\ |K_r|_1 &\leqslant \kappa_2^r \cdot r! \\ |\Sigma_r|_1 &\leqslant \sigma_2^r \cdot r! \\ |G_{2m,r}|' &\leqslant \Gamma_{2m}^r \cdot r! \end{aligned} \tag{1.50}$$

Thus, to all orders in λ, the Green functions of the model with one-particle band structure $e + K$ can be calculated in renormalized perturbation theory, and they are given by almost-everywhere finite functions of the independent external momenta (the momentum conservation delta function is already taken out). The self-energy is a continuously differentiable function of p_0 and $\mathbf{p}$; the counterterms K_r are finite and continuously differentiable in $\mathbf{p}$. The counterterms K^I_r are constructed recursively in r ("order by order in the expansion in λ") from (2.76); the diagrams that contribute are of self-energy type. Since the amputated function G_2 is first order in λ, i.e., the free propagator is subtracted from the two-point function before amputating to get G_2, the unamputated connected two-point function indeed tends to $(ip_0 - e(\mathbf{p}) - \Sigma(p))^{-1}$ in the limit $I \to -\infty$. Thus Σ is the usual (Dyson) self-energy. Because $\Sigma(0, \mathbf{p}) = (\ell\Sigma)(p) = 0$ for all $\mathbf{p} \in S$, the interacting model with one-particle band structure $e + K$ has the same Fermi surface at the given value μ of the chemical potential as the free model with band structure e. In other words, the effect of renormalization is indeed that the interacting Fermi surface is kept fixed. This is a much more delicate condition than in the spherical case, where the function K reduces to a constant, i.e., a shift $\delta\mu$ in the chemical potential.

The infrared limit of the renormalized expansion gives the same convergent Green functions if we choose a finite volume and positive temperature, and the same conclusions hold with functions that have a limit as the volume tends to infinity and/or the temperature goes to zero. The point of the renormalization in finite volume and at finite temperature, where there are no divergences in the loop integrals, is to rearrange the expansion in a way that uniformity in volume and temperature and convergence of the expansion coefficients for the Green functions in the thermodynamic and zero-temperature limit is achieved. Of course, by the above discussion, this rearrangement amounts precisely to keeping track of how the Fermi surface moves when the interaction is turned on.

If the bound for the coefficients (1.50) is saturated, the renormalized expansion has convergence radius zero, which in itself may not seem a very useful statement. However, if one is willing to go to a slightly more technical level and consider the representation of the Green functions as sums over values of Feynman graphs, the renormalization method also yields much more precise and detailed statements about when and why the series diverges. It is a well-known fact in renormalizable field theories that the only source of factorial growth of individual diagrams is the marginal scale behavior of insertions of four-legged subdiagrams. In this paper we show the stronger statement that if there are no *ladder* subdiagrams, the values of all graphs are bounded without the r factorial, i.e., we specify the set of those four-legged diagrams that can really produce factorials much more precisely. The meaning of the statement that in a given graph there are no ladder subdiagrams is defined in Section 2.5: they are the graphs that contain no four-legged nonoverlapping subdiagrams to any scale. This statement is useful because the structure of these of these graphs is given explicitly in Section 2.4. The four-legged nonoverlapping diagrams are ladder diagrams, also called bubble chains, where the fermion lines may be dressed with Hartree–Fock-type corrections. However, any vertex corrections or polarization subdiagrams make the graphs overlapping and its scale sum convergent instead of marginally divergent. The detailed bound is stated and discussed in Section 2.7; it also depends on the tree decomposition of the graph. A short version is as follows.

Theorem 1.3. Let G be a graph contributing to $G_{2m,r}$, and denote by $V(G)$ the norm of the scale sum of $\mathrm{Val}(G^J)$, where J is any labeling of G, and the norm depends on the number of external legs, as in Theorem 1.2. If for any labeling J, G^J does not contain any nonoverlapping four-legged subdiagrams at any scale, then $V(G) \leqslant V_m^r$, where V_m is a constant independent of r.

In other words, Theorem 1.3 means that a single graph in the nth order of perturbation theory can have value $\sim n!$ only if it contains ladder subdiagrams. All other four-legged insertions do not produce any factorials in the value of single graphs. This suggests that only insertions of ladder diagrams can change the behavior of the correlations, i.e., the properties of the ground state, qualitatively, and that all other corrections are analytic in the coupling. A nonperturbative proof of this requires control over the sum of all graphs, hence an implementation of the Pauli principle, which has been done in $d=2$ spatial dimensions.[5]

In the case of a strictly convex Fermi surface, only the behavior at transfer momentum zero can lead to factorials in the values of individual graphs, because at all other values of the momentum, the surface intersects transversally with its translate or at least there is a curvature effect that implies the absence of a singularity. In the spherical case the existence of a singularity at zero transfer momentum has been shown to be responsible for the occurrence of off-diagonal long-range order in the ground state.[3,5] If the Fermi surface is transversal to its negative, the ladder graphs are nonsingular and analyticity in λ holds in infinite volume.[7]

The classification of graphs into overlapping and nonoverlapping ones that will be introduced in Section 2.4 may seem technical at first; it is, however, natural since the graphs that are nonoverlapping to all scales are the dressed ladder graphs in the four-legged case and the Hartree–Fock graphs in the two-legged case. The four-legged nonoverlapping graphs are the only ones that do not show improved power counting behavior, and in this sense their resummation is a resummation of the "leading divergences."

Note, however, that Theorem 1.3 is a statement about the behavior of values of single graphs, and does not require any resummation. Therefore it holds irrespective of the sign of the coupling (on which the existence of solutions to the gap equations from resummation depends). Also, it holds for the general class of nonflat Fermi surfaces given by our Assumptions A1–A3, and not just for strictly convex Fermi surfaces.

It is technically necessary to do an expansion with a fixed interacting Fermi surface, to prevent the problems described above when one expands a moving singularity in terms of a fixed one. In order to construct a model with a given one-particle band structure and to see how the Fermi surface moves under the interaction, and also to clarify the relation between the counterterm function K and the self-energy, we have to study the map $e \mapsto E = e + K$ further and show that it is invertible. To invert this map, one would like to take a derivative of K with respect to e. It is not obvious that such a derivative exists since K is a functional of e obtained by integrating factors of $1/(ip_0 - e(\mathbf{p}))$, and taking a derivative produces a square of the denominator, and thus potentially a singularity, since the square of the propagator is not locally integrable. However, the volume improvement bounds allow us to take this derivative. The latter is also necessary to get information about the dependence on the chemical potential μ, since the expansion has so far been done at a fixed value of μ, which then fixes the Fermi surface. Different values of μ give rise to different Fermi surfaces, and in the case without spherical symmetry, different also means of different shape. The renormalization would be useless if it worked only pointwise in μ. In other words, it is important to establish some continuity properties in μ. We show that the counterterms and thus the self-energy and the value of any graph are continuously differentiable in μ. More generally, we prove that this C^1 property holds for derivatives with respect to e, i.e., we allow for much more general variations of the band structure than just a shift by a constant. Let

$$D_h K_r^I(e, h) = \frac{\partial}{\partial\alpha} K_r^I(e + \alpha h)\bigg|_{\alpha=0} \tag{1.51}$$

be the directional derivative of K_r^I with in direction h.

Theorem 1.4. If Assumptions A1–A3 hold, then $\lim_{I \to -\infty} D_h K_r^I(e, h)$ exists for all $r \geqslant 1$ and

$$\left| \lim_{I \to -\infty} D_h K_r^I(e, h) \right|_0 \leqslant \text{const}(r)\, |h|_0 \tag{1.52}$$

Corollary 1.5. If Assumptions A1–A3 hold, then the counterterms K are continuously differentiable functions of the chemical potential μ.

To convert this statement about directional derivatives into one about derivatives as bounded linear operators[8] and to consider varying e, not just μ, we have to be more specific about the set of allowed e's. Let $\emptyset \neq \mathcal{N} \subset \mathcal{B}$ be open. For $k \geqslant 0$, denote the Banach spaces $(C^k(\mathcal{B}, \mathbb{R}), |\cdot|_k)$ by $\mathcal{C}^k$ and $(C^k(\bar{\mathcal{N}}, \mathbb{R}), |\cdot|_k)$ by $\mathcal{C}^k_{\mathcal{N}}$. For $1 \leqslant \sigma \leqslant d-1$, $g_2 > g_0 > 0$, and $g_3 > 0$ let

$$\begin{aligned} &\mathcal{A}_2(\sigma, \mathcal{N}, g_0, g_2, g_3) \\ &\quad = \Big\{ e \in \mathcal{C}^2_{\mathcal{N}} : |e|_2 < g_2,\ S(e) = \{\mathbf{p} \in \mathcal{B} : e(\mathbf{p}) = 0\} \subset \mathcal{N}, \\ &\qquad |\nabla e(\mathbf{p})| > g_0 \text{ for all } \mathbf{p} \in \mathcal{N}, \text{ and } n: S(e) \to S^d, \\ &\qquad \omega \mapsto n(\omega) = \frac{\nabla e}{|\nabla e|}(\omega) \text{ satisfies: for all } \omega \in S, \\ &\qquad \text{rank } dn(\omega) \geqslant \sigma, \text{ and all nonzero eigenvalues } m \\ &\qquad \text{of } dn \text{ satisfy } |m| > g_3 \Big\} \end{aligned} \tag{1.53}$$

Here $S^d = \{a \in \mathbb{R}^d : |a| = 1\}$, and dn is the derivative of n with respect to $\omega \in S$. In other words, dn is the derivative of n tangential to the surface S

[note that $dn(\omega)$ is a quadratic matrix since the dimension of $S(e)$ is $d-1$, and $n(\omega) \in S^d$]. Let $\mathscr{L}$ be the space of bounded linear operators from $\mathscr{C}^2_{\mathcal{N}}$ to $\mathscr{C}^0_{\mathcal{N}}$.

Theorem 1.6. Let $1 \leqslant \sigma \leqslant d-1$ and $g_2 > g_0 > 0$. Then $\mathscr{A} = \mathscr{A}_2(\sigma, \mathcal{N}, g_0, g_2)$ is open in $\mathscr{C}^2_{\mathcal{N}}$. For all $e \in \mathscr{A}$, Assumptions A2 and A3 hold, with $\kappa = \sigma$ and $\rho = 1$. For all $e \in \mathscr{A}$ and all $r \geqslant 1$, $D_h K_r^I(e, h) = (K_r^I)'(e) h$ with $(K_r^I)'(e) \in \mathscr{L}$, and there is $K_r' \in \mathscr{L}$ such that

$$\|(K_r^I)'(e) - K_r'\|_{\mathscr{L}} \to 0 \qquad \text{as} \quad I \to -\infty \tag{1.54}$$

The function $K_r \colon \mathscr{A} \to \mathscr{C}^0_{\mathcal{N}}$ is differentiable in e, and its derivative is given by $K_r' \in \mathscr{L}$. The map $e \mapsto K_r'(e)$ is continuous on $\mathscr{A}$, and there is $C_r > 0$ such that for all $h \in \mathscr{C}^2_{\mathcal{N}}$

$$|K_r'(e) h|_0 \leqslant C_r \, |h|_0 \tag{1.55}$$

C_r is independent of $e \in \mathscr{A}$.

The bound (1.55) is the most subtle result of this paper. Note that no derivative acts on h on the right side of (1.55). Because of that, K_r' extends uniquely to a bounded linear operator on $\mathscr{C}^0_{\mathcal{N}}$, and we can prove the following.

Theorem 1.7. Let $R \geqslant 1$, $\lambda \in \mathbb{R}$, and for $e \in \mathscr{A}$, let

$$E_\lambda^{(R)}(e) = e + \sum_{s=1}^{R} \lambda^s K_s(e)$$

If

$$\sum_{r=1}^{R} C_r \, |\lambda|^r < 1 \tag{1.56}$$

then $E_\lambda^{(R)}$ is injective on every convex subset of $\mathscr{A}$. That is, if $e_1, e_2 \in \mathscr{A}$ with $E_\lambda^{(R)}(e_2) = E_\lambda^{(R)}(e_2)$, and if $(1-s) e_1 + s e_2 \in \mathscr{A}$ for all $s \in [0, 1]$, then $e_1 = e_2$.

Since $\mathscr{A}$ is open, the maximal ball around any $e \in \mathscr{A}$ is such a convex subset. Thus $E_\lambda^{(R)}$ is locally injective. The significance of Theorem 1.7 for the problem of self-consistent renormalization is discussed in the next section.

The set $\mathscr{A}$ of Theorem 1.6 is more restricted than the set of all e satisfying Assumption A3. It comprises the case of a strictly convex Fermi surface, or that of a torus or a cylinder, but, e.g., not Example 3 of Section 1.3. More generally, the specification of a set of e for which $dn(\omega)$ may vanish for some ω on S, but for which the exponent $\rho \geqslant 1$ is still uniform in e, requires the existence of more derivatives ($k > 2$) of e. This can be formulated, but for conciseness, we restricted consideration to the simplest case here. The reader may construct his or her own generalizations; the essential requirement is that the constants u_0 (defined in Section 2) and the volume improvement exponent ε must be uniform on the set. The reason we gave A3 as a separate assumption is that it is more general and can be checked without trouble in examples; for instance, it is easy to see that for Example 3, $e(\mathbf{p}) = p_1^{2n} + p_2^{2n} - \mu$, there is an open μ-interval with the desired properties, and this already suffices to prove Theorems 1.2–1.5.

Finally, we define the Hartree–Fock aproximation as the sum over all graphs that are nonoverlapping on all scales; equivalently, these are the graphs produced by iterating the Hartree–Fock integral equation (1.40). This resummation also defines a map $e \mapsto e + H$, where H represents the counterterms in the Hartree–Fock approximation.

Theorem 1.8. The map $e \mapsto e + H$ is invertible in every fixed order in perturbation theory.

This theorem is easy to prove; we shall discuss its motivation in the next section.

1.6. Discussion

The interpretation of renormalization is thus: the unrenormalized Green functions diverge because it is wrong to assume that both the band structure and the Fermi surface stay fixed when the interaction λV is turned on. In reality, they respond to the interaction—if the surface is fixed, the band structure changes, and vice versa (this is similar to the situation in KAM theory, where the frequencies and actions of quasiperiodic orbits cannot both stay fixed under a perturbation). To do the expansion, we prefer not to let the Fermi surface move, since the moving of the singularity produces the divergences discussed above. Instead we allow for a change in the band structure e. The function $K(\mathbf{p})$ contains the terms that are necessary to prevent the surface from moving under the perturbation. This function depends on the vector field u which we used to define the projection onto the Fermi surface. The dependence on u amounts to a reparametrization of the Fermi surface and has no physical consequences.

It has long been known in solid-state theory that self-energy effects have to be taken into account to avoid divergences in perturbation theory. In many accounts this is described as self-consistent renormalization, with the idea that if the free two-point function is expressed in terms of the exact two-point function everywhere, two-legged insertions disappear, since they arose from self-energy terms. This procedure is usually called "self-consistent" renormalization and described in words in the literature. Often, one then goes on to describe particular approximations, such as the Hartree–Fock approximation. Since none of these approximations is exact, none of them removes all the divergences, and one point of our analysis is that we give a clear procedure how to do this to all orders in perturbation theory: the divergences are removed by fixing the Fermi surface. Self-consistent renormalization is then achieved by inversion, i.e., solving the equation

$$E = e + K(e, \lambda) \tag{1.57}$$

for e in terms of a given E. It is really a separate step. Once this is done, the combination of renormalization and inversion allows one to determine how the Fermi surface moves when the band structure is fixed.

Obviously, the solution of (1.57) requires some knowledge of the regularity properties of the map K, which is a map between function spaces. We have established enough regularity to show that $id + K$ is locally injective. In other words, we have proven that for any interacting band structure, there is locally at most one bare band structure that has the same Fermi surface, i.e., uniqueness of the solution. The existence (surjectivity) proof requires more detailed bounds and more stringent assumptions and will appear in a sequel paper.

Note that this regularity problem does not just arise because we use counterterms to do the expansion correctly. Any attempt to "consider only skeleton graphs first and then replace the free propagator by the interacting one on all lines" also requires the inversion of the map $e \mapsto e + \Sigma$, and the regularity problem is thus similar to ours (only harder, since Σ also depends on p_0). In brief, in any way of looking at the system, there is the question of how regular the self-energy, and thus the interacting Fermi surface, is. In the heuristic discussion after (1.40) that motivated why the divergences are artificial, this question was postposed by the assumption that Σ is "reasonable", so that (1.36) can be used to show well-definedness of the left side of (1.41). However, Σ is not a function one is free to choose or make assumptions about. It is determined by the iteration, and therefore its regularity has to be proven. We have proven that Σ and K are C^1 (Theorem 1.2). Inverting (1.57) requires at least $K \in C^2$. The proof of this will appear in another paper.

Regularity of the Hartree–Fock approximation to Σ (Theorem 1.8) is easily shown: it is obvious that the external momentum can be routed through an interaction line in every Hartree–Fock graph, so the Hartree–Fock self-energy has the same regularity properties as the interaction potential.

Since improved power counting plays a central role in the technical analysis done here and since the facts on which it is based are not specific to our multiscale analysis and therefore have wider applications, we

Fig. 5. (a) A shell around the Fermi surface, (b) transversal intersection with a translate, and (c) nontransversal intersection with a translate.

describe briefly how it comes about. A way to understand power counting is to weight the growth of the propagator in the vicinity of its singularity S against the smallness of the volume of shells around the Fermi surface, where it becomes large. We use a scale decomposition where momentum space is cut into shells around the Fermi surface, as sketched in Fig. 5a. It is easy to see that the $\mathbf{p}$-volume of a shell in which (say) $2^{j-1} \leqslant |e(\mathbf{p})| \leqslant 2^j$ $(j<0)$ is bounded by a constant times 2^j (see also Section 2.1). It is also easy to deduce the integrability properties of C that we discussed above by weighting this volume against the growth of $|C|$ in a summation over shells:

$$\begin{aligned}
\int_{|ip_0 - e(\mathbf{p})| \leqslant 1/2} dp_0 \frac{1}{|ip_0 - e(\mathbf{p})|^\alpha}
&= \sum_{j<0} \int_{\mathbb{R}} dp_0 \int_{\mathscr{B}} d\mathbf{p} \frac{1(2^{j-1} < |ip_0 - e(\mathbf{p})| \leqslant 2^j)}{|ip_0 - e(\mathbf{p})|^\alpha} \\
&\leqslant \sum_{j<0} 2^{(1-j)\alpha} \int_{-2^j}^{2^j} dp_0 \int_{\mathscr{B}} d\mathbf{p}\, 1(2^{j-1} < |e(\mathbf{p})| \leqslant 2^j) \\
&\leqslant \text{const} \cdot 2^\alpha \sum_{j<0} 2^{-j\alpha} \cdot 2^j \cdot 2^j \\
&= \text{const} \cdot 2^\alpha \sum_{j<0} 2^{j(2-\alpha)}
\end{aligned} \tag{1.58}$$

which converges if $\alpha < 2$. Up to this point, this is just a rewriting of (1.36). However, the geometry of these shells has important consequences for nontrivial graphs, which we discuss now.

Beyond lowest order, the momentum assignments in graphs with at least two lines consist of linear combinations of the loop momenta with the external momenta, e.g., $\mathbf{p}$ and $\mathbf{p}+\mathbf{q}$, where $\mathbf{p}$ is a loop momentum. On scale j, both $\mathbf{p}$ and $\mathbf{p}+\mathbf{q}$ must be in a shall of thickness 2^j around S. The volume of the full shell is of order 2^j. On the other hand, for most values of $\mathbf{q} \in \mathscr{B}$, the intersection of S and $\pm S + \mathbf{q}$ will be transversal. Thus the support of the integrand will have a volume which is much smaller, roughly by a factor 2^j, since the volume is no longer that of an entire shell, but that of a transversal intersection of two shells around S (see Fig. 5b). However, for those values of $\mathbf{q}$ where the intersection of the shell with its translate by $\mathbf{q}$ is not transversal, e.g., for $\mathbf{q}=0$, or the translation shown in Fig. 5c, there is no gain at all, i.e., there is no uniformity in $\mathbf{q}$. The improved power counting bound is based on the observation that if there is no nesting in the sense that A3 holds, then the set of $\mathbf{q}$ for which the intersection is not transversal has small volume itself. So, if $\mathbf{q}=\mathbf{k}+\mathbf{Q}$, where $\mathbf{k}$ is another loop momentum, and if there was no gain in the integration over $\mathbf{p}$, $\mathbf{k}$ must be in a set of small volume. This restriction produces an additional "volume improvement factor" $2^{\varepsilon j}$ in the second loop integration over $\mathbf{k}$ (this also applies to the surface drawn in Fig. 5). Thus, in the double integral over $\mathbf{p}$ and $\mathbf{k}$ that appears in (1.34), there will *always* be an improvment factor $2^{\varepsilon j}$, which is uniform in $\mathbf{Q}$. Therefore, $\mathbf{Q}$ may be an arbitrary combination of loop momenta and external momenta, and it is not necessary to keep track of all complications of the momentum flow in general graphs to extract the improvement factor. It is only necessary to find out which graphs have this volume gain, i.e., contain a factor I_2 as a subintegral. Obviously, they must have at least two loops, but the above condition that $\mathbf{q}=\mathbf{k}+\mathbf{Q}$ with another loop momentum $\mathbf{k}$ means also that there must be a fermion line in which two loop momenta flow (the two loop momenta $\mathbf{p}$ and $\mathbf{k}$ flow in the line with momentum $\mathbf{p}+\mathbf{q}=\mathbf{p}+\mathbf{k}+\mathbf{Q}$). This is now a purely graph-theoretic question. The class of graphs for which such a line exists is precisely that of overlapping graphs defined in Section 2.4. The nonoverlapping graphs are classified explicitly in the two- and four-legged cases (it is not hard to generalize the characterizations given in Section 2.4 to graphs with more that four external legs, but we do not need that here). The volume improvement bound (1.33) is proven under the hypotheses A2 and A3 in Appendix A by the argument outlined above.

Note that the above transversality and no-nesting arguments require $d\geqslant 2$. In $d=1$, improved power counting is absent. This is one reason why one-dimensional many-fermion models behave differently from the higher dimensional ones.

The proofs in Section 2.4 are elementary and independent of the scale decomposition. Indeed, the only property of the model that is used in Section 2.4 is that all vertices have an even incidence number, which is true in our class of models since the interaction is a four-fermion interaction (see also Fig. 4 in Section 2.4). For vertices with an odd incidence number, a similar classification can be done.

The implementation of these graphical statements for the volume gains in the problem with the full scale structure is done in Sections 2.5 and 2.6

and used thereafter to prove the stated theorems. Some of these proofs are not short, but they are in principle an application of the simple ideas stated above.

2. RENORMALIZATION AND CONVERGENCE

In this section, we set up the renormalization flow and define the localization operator that is used to subtract the value of two-legged diagrams on the Fermi surface. We then develop one of the main technical tools, the graph structure lemmas that are used to extract volume improvement factors systematically for any labeled graph. We use this to show an improved power counting bound, and then show that the renormalized Green functions converge in every order in perturbation theory, and that the only four-legged graphs which do not obey improved power counting are the ladder graphs.

We start with some elementary remarks that follows from the assumptions. By A2, S is a compact $(d-1)$-dimensional C^k-submanifold of $\mathscr{B}$. Let

$$U_\eta(S) = \{\mathbf{p}\colon \exists \mathbf{q} \in S \text{ with } |\mathbf{p}-\mathbf{q}| < \eta\} \tag{2.1}$$

Then there is δ such that $G_0 = \sup\{|\nabla e(\mathbf{p})|\colon \mathbf{p} \in U_{2\delta}(S)\}$ is finite, and such that $g_0 = \inf\{|\nabla e(\mathbf{p})|\colon \mathbf{p} \in U_{2\delta}(S)\} > 0$. Let u be a vector field on a neighborhood $U_\delta(S)$ of S. We call u transversal to S if there is $u_0 > 0$ such that for all $\mathbf{p} \in S$, $\nabla e(\mathbf{p}) \cdot u(\mathbf{p}) \geqslant u_0 > 0$. Denote the integral curve of u passing through $\mathbf{p} \in S$ by $\gamma_\mathbf{p}$, that is, $\gamma_\mathbf{p}\colon (-t_0, t_0) \to \mathscr{B}$, $t \mapsto \gamma_\mathbf{p}(t)$, $\gamma_\mathbf{p}(0) = \mathbf{p}$, and for all $t \in (-t_0, t_0)$, $(\partial/\partial t)\, \gamma_\mathbf{p}(t) = u(\gamma_\mathbf{p}(t))$.

Lemma 2.1. Assume A2.

(i) There is a C^∞ vector field u transversal to S, and there is $t_0 > 0$ such that $\Psi\colon S \times (-t_0, t_0) \to \Psi(S \times (-t_0, t_0)) \subset \mathscr{B}$, defined by $\Psi(\mathbf{p}, t) = \gamma_\mathbf{p}(t)$, is a C^k-diffeomorphism.

(ii) There are $\delta > 0$ and $u_0 \in (0, 1)$ such that $\overline{U_{2\delta}(S)} \subset \Psi(S \times (-t_0, t_0))$, and such that for all $\mathbf{q} \in U_{2\delta}(S)$: $0 < g_0/2 \leqslant u_0 \leqslant \nabla e(\mathbf{q}) \cdot u(\mathbf{q}) \leqslant G_0$.

(iii) Define the functions $\tau\colon \overline{U_{2\delta}(S)} \to \mathbb{R}$ and $\mathbf{P}\colon \overline{U_{2\delta}(S)} \to S$ as follows. For $\mathbf{q} \in \overline{U_{2\delta}(S)}$

$$(\mathbf{P}(\mathbf{q}), \tau(\mathbf{q})) = \Psi^{-1}(\mathbf{q}) \tag{2.2}$$

In other words, $\gamma_{\mathbf{P}(\mathbf{q})}(\tau(\mathbf{q})) = \mathbf{q}$. Then

$$\mathbf{q} = \mathbf{P}(\mathbf{q}) + \int_0^{\tau(\mathbf{q})} u(\gamma_{\mathbf{P}(\mathbf{q})}(t))\, dt \tag{2.3}$$

so $|\mathbf{q}-\mathbf{P}(\mathbf{q})| \leqslant |\tau(\mathbf{q})|$ and

$$|\mathbf{q}-\mathbf{P}(\mathbf{q})| \leqslant \frac{1}{u_0} |e(\mathbf{q})| \tag{2.4}$$

Furthermore, $u_0 \leqslant e(\mathbf{q})/\tau(\mathbf{q}) \leqslant G_0$.

(iv) Let $\mathbf{p} \in U_\delta(S)$, $\rho = e(\mathbf{p})$, and $\omega = \mathbf{P}(\mathbf{p})$. The map $\chi: \mathbf{p} \mapsto (\rho, \omega)$ is a C^k-diffeomorphism from $U_\delta(S)$ to a subset of $\mathbb{R} \times S$. Denoting its inverse map by $\mathbf{p}(\rho, \omega)$, there are constants A_0 and A_1 such that the Jacobian $J(\rho, \omega) = \det \mathbf{p}'(\rho, \omega)$ obeys

$$\sup_{\mathbf{p} \in U_\delta(S)} |J(\rho, \omega)| \leqslant \frac{1}{u_0} A_0 \tag{2.5}$$

and its derivative ∂J obeys

$$\sup_{\mathbf{p} \in U_\delta(S)} |\partial J(\rho, \omega)| \leqslant \frac{1}{u_0^2} A_1 \tag{2.6}$$

A_0 depends on δ, u_0, and $|u|_1$; A_1 also depends on the second derivative of u.

Proof. (i, ii) We show that $u \in C^\infty$ transversal to S exists even if e is only C^1. For $\mathbf{p} \in U_{\delta_0}(S)$, let $n(\mathbf{p}) = \nabla e(\mathbf{p})/|\nabla e(\mathbf{p})|$; then n is continuous in $\mathbf{p}$. So for all $\mathbf{p} \in S$ there is $r(\mathbf{p}) > 0$ such that $n(\mathbf{p}) \cdot n(\mathbf{p}') > 1/2$ for all $\mathbf{p}' \in U_{2r(\mathbf{p})}(\mathbf{p})$. Since S is compact, the covering $(U_{r(\mathbf{p})}(\mathbf{p}))_{\mathbf{p} \in S}$ contains a finite subcovering by $V_i = U_{r(\mathbf{p}_i)}(\mathbf{p}_i)$, $i \in \{1,..., n\}$, and there is $\delta > 0$ such that $U_{3\delta}(S) \subset \bigcup_{i=1}^n V_i$. Now, $U_{3\delta}(S) \subset \mathscr{B}$ is open, hence a C^∞ submanifold of $\mathscr{B}$. Choose a C^∞ partition of unity $(\chi_i)_i$ that is subordinate to the cover $V_i \cap U_{3\delta}(S)$ and that obeys $\operatorname{supp} \chi_i \subset U_{r(\mathbf{p}_i)}(\mathbf{p}^i)$ for all i. Define

$$u(\mathbf{p}) = \sum_{i=1}^{n} \chi_i(\mathbf{p}) \, n(\mathbf{p}_i) \tag{2.7}$$

Then $u \in C^\infty(U_{3\delta}(S), \mathbb{R}^d)$, and by construction of χ_i,

$$\begin{aligned} u(\mathbf{p}) \cdot \nabla e(\mathbf{p}) &= \sum_{i:\, \mathbf{p} \in U_{r(\mathbf{p}_i)}(\mathbf{p}_i)} \chi_i(\mathbf{p}) \, |\nabla e(\mathbf{p})| \, n(\mathbf{p}_i) \cdot n(\mathbf{p}) \\ &\geqslant \frac{g_0}{2} \sum_i \chi_i(\mathbf{p}) = \frac{g_0}{2} \end{aligned} \tag{2.8}$$

Part (i) is now obvious since S in a C^k-submanifold of $\mathscr{B}$, and (ii) is clear by construction of u.

(iii) Equation (2.3) holds by definition of the map Ψ and that of the integral curve γ. It obviously implies

$$|\mathbf{q}-\mathbf{P}(\mathbf{q})| \leqslant \left| \int_0^{\tau(\mathbf{q})} u(\gamma_{\mathbf{P}(\mathbf{q})}(t))\, dt \right| \leqslant |\tau(\mathbf{q})| \tag{2.9}$$

since $|u| \leqslant 1$. If $e(\mathbf{q}) \geqslant 0$, then $\tau(\mathbf{q}) \geqslant 0$ and, since $e(\mathbf{P}(\mathbf{q}))=0$,

$$\begin{aligned} e(\mathbf{q}) &= \int_0^{\tau(\mathbf{q})} \frac{d}{dt} e(\gamma_{\mathbf{P}(\mathbf{q})}(t))\, dt \\ &= \int_0^{\tau(\mathbf{q})} (u \cdot \nabla e)(\gamma_{\mathbf{P}(\mathbf{q})}(t))\, dt \geqslant u_0 \tau(\mathbf{q}) \geqslant u_0\, |\mathbf{q}-\mathbf{P}(\mathbf{q})| \end{aligned} \tag{2.10}$$

The case $e(\mathbf{q}) \leqslant 0$ is similar.

(iv) The map is a diffeomorphism because it is the composition of Ψ with the inverse of $(\omega, \tau) \mapsto (\omega, \rho) = (\omega, e(\Psi(\omega, \tau)))$ and because

$$\frac{\partial \rho}{\partial \tau}(\omega, \tau) = (\nabla e \cdot u)(\Psi(\omega, \tau)) \tag{2.11}$$

so that $\partial\rho/\partial\tau \geqslant u_0$ in $\Psi^{-1}(U_\delta(S))$. This also implies the bounds for the Jacobian and its derivative. ∎

Remark 2.2. The choice $u = \nabla e/|\nabla e|$ has most of the above properties, with $u_0 = g_0$, but it is only C^{k-1} if e is C^k, and then the maps Ψ and χ are only C^{k-1}. In particular, with this choice of u, finiteness of A_1 requires $k \geqslant 3$ in A2.

2.1. Scale Decomposition and Power Counting

Let ε be as in A3, u_0 and δ as in Lemma 2.1, and $M > \max\{4^{1/\varepsilon}, 1/(u_0\delta)\}$. Then $|e(\mathbf{p})| \leqslant M^{-1}$ implies $\mathbf{p} \in U_\delta(S)$. Let $a \in C^\infty(\mathbf{R}_0^+, [0, 1])$ be such that

$$a(x) = \begin{cases} 0 & \text{for } x \leqslant M^{-4} \\ 1 & \text{for } x \geqslant M^{-2} \end{cases} \tag{2.12}$$

and $a'(x) > 0$ for all $x \in (M^{-4}, M^{-2})$. Set

$$f(x) = a(x) - a(x/M^2) = \begin{cases} 0 & \text{if } x \leqslant M^{-4} \\ a(x) & \text{if } M^{-4} \leqslant x \leqslant M^{-2} \\ 1 - a(x/M^2) & \text{if } M^{-2} \leqslant x \leqslant 1 \\ 0 & \text{if } x \geqslant 1 \end{cases} \tag{2.13}$$

so that, for all $x>0$, $f(x)\geqslant 0$ and

$$1-a(x)=\sum_{j=-\infty}^{-1} f(M^{-2j}x) \tag{2.14}$$

Calling $f_j(x)=f(M^{-2j}x)$, we have

$$\operatorname{supp} f_j=[M^{2j-4}, M^{2j}] \tag{2.15}$$

and for all $x\geqslant 0$,

$$f_j(x)\,f_{j'}(x)=0 \qquad \text{if} \quad |j-j'|\geqslant 2 \tag{2.16}$$

Defining

$$C_j(x, y)=\frac{f(M^{-2j}\,|x+iy|^2)}{ix-y}=\frac{f_j(|x+iy|^2)}{ix-y} \tag{2.17}$$

we decompose

$$\frac{e^{ip_0 0^+}}{ip_0-e(\mathbf{p})}=\frac{e^{ip_0 0^+}a(p_0^2+e(\mathbf{p})^2)}{ip_0-e(\mathbf{p})}+e^{ip_0 0^+}\sum_{j<0} C_j(p_0, e(\mathbf{p})) \tag{2.18}$$

For the purpose of the present paper, we discard the ultraviolet end of the model by removing the first term in this sum, in other words, taking $(1-a)/(ip_0-e(\mathbf{p}))$ as propagator. The infrared singularity, which is the physically relevant feature of the problem, is unchanged.

Lemma 2.3. For all $j<0$:

(i) $|C_j|_0\leqslant M^{-j+2}$. More precisely, for all $p=(p_0, \mathbf{p})$,

$$|C_j(p_0, e(\mathbf{p}))|\leqslant M^{-j+2}1(|ip_0-e(\mathbf{p})|\in[M^{j-2}, M^j]) \tag{2.19}$$

(ii) Let A_0 be as in Lemma 2.1 and $A=\max\{1, 2A_0\int_S d\omega\}$. Then

$$\int_{\mathscr{B}} d^d\mathbf{p}\ \ 1(|e(\mathbf{p})|\leqslant\eta)\leqslant\frac{A}{u_0}\eta \tag{2.20}$$

and

$$|C_j|'=\int_{\mathbb{R}\times\mathscr{B}} dp_0\,d\mathbf{p}\,|C_j(p_0, e(\mathbf{p}))|\leqslant\frac{2AM^2}{u_0}M^j \tag{2.21}$$

In particular, taking

$$K_0 = \frac{2AM^2}{u_0} \tag{2.22}$$

we have

$$|C_j|_0 \leqslant K_0 M^{-j} \qquad \text{and} \qquad |C_j|' \leqslant K_0 M^j.$$

(iii) For any multiindex α with $s = |\alpha| \leqslant k$, there is a constant W_s depending on $|e|_s$ and M such that

$$|D^\alpha C_j(p_0, e(\mathbf{p}))| \leqslant W_s M^{-(s+1)j} 1(|ip_0 - e(\mathbf{p})| \in [M^{j-2}, M^j]) \tag{2.23}$$

The proof of this lemma is easy; we leave it as an exercise to the reader. This lemma implies that for any $0 > I > -\infty$, any power of the propagator $\sum_{j \geqslant I} C_j$ is integrable and so values of connected graphs evaluated according to the above Feynman rules, but with this cutoff propagator instead of $1/(ip_0 - e(\mathbf{p}))$, are finite, and C^∞ in p_0 and e.

The bounds given in the lemma are similar to those in the spherical case,[2,3] and so the power counting is the same as in Lemma III.1 of ref. 3. The dimension δ_l [see ref. 3, Eq. (III.5)] is $\delta_l = 1$.

We now state the analogue of the abstract power counting lemma of ref. 3. For the moment, we refer the reader to ref. 3 for details about labeled graphs and the associated trees; they will be explained in more detail in Section 2.3. Let G be a connected graph with an even number E of external lines, and two- and four-legged vertices. Let $L(G)$ be the set of internal lines of G and J: $L(G) \to \{z \in \mathbb{Z}: 0 > z \geqslant I\}$, $l \mapsto j_l$ be a labeling of G, which assigns a scale to each line of G. Construct the tree $t = t(G^J)$ associated to the labeled graph G^J as follows.[3] The forks f of the tree are the connected components G^J_f of all the subgraphs $\{l \in L(G^J)\colon j_l \geqslant h\}$ with $h \leqslant -1$. The subgraphs are partially ordered by inclusion to form $t(G^J)$. The scale of a fork is defined by

$$j_f = \min\{j_l\colon l \in L(G^J_f)\}$$

Define, in analogy to Eqs. (III.9) and (III.10) of ref. 3,

$$\begin{aligned}
D_f &= |L(G^J_f)| - 2(|V(G^J_f)| - 1) \\
\Delta_f &= -\tfrac{1}{2}|\{l\colon l \text{ internal line of } G^J,\ l \text{ external line of } G^J_f\}| \\
\Delta_v &= -\tfrac{1}{2}|\{l\colon l \text{ internal line of } G^J,\ v \in l\}|
\end{aligned} \tag{2.24}$$

where $V(G)$ is the set of vertices of G. The value of the graph G^J is defined as

$$\delta^{\#}(p_{\text{out}}-p_{\text{in}})\, Val(G^J)=\sum_{\text{spins }\alpha}\int\prod_{j\in L(G)} d^{d+1}p_l\, C_{j_l}(p_l)\,\delta_{\alpha_l\alpha_l'}$$
$$\times\prod_{v\in V_4(G)}\delta^{\#}(p_{\text{out},v}-p_{\text{in},v})(\mathscr{U}_v)_{\alpha_1^{(v)}\dots\alpha_4^{(v)}}(q_v)$$
$$\times\prod_{v\in V_2(G)}\delta^{\#}(p_{\text{out},v}-p_{\text{in},v})\,\theta_v(q_v) \tag{2.25}$$

where $L(G)$ is the set of internal lines of G, $V_4(G)$ is the set of four-legged vertices of G and $V_2(G)$ is the set of two-legged vertices of G, $\theta(q_v)$ is the function associated to a two-legged vertex v, $\mathscr{U}$ is the vertex function associated to a four-legged vertex v, and the momenta q_v are given in terms of external and loop momenta by the momentum conservation at every vertex. The spin indices on a line l and a vertex v are the same if l goes into v or out of v, and the symbol sum over spins indicates that they are summed over.

Lemma 2.4. Let K_0 be as in Lemma 2.3. Then

$$|Val(G^J)|_0\leqslant(4K_0)^{|L(G)|}\prod_{v\in V_2(G)}|\theta_v|_0\prod_{v\in V_4(G)}|\mathscr{U}_v|_0\, M^{D_\phi j_\phi}$$
$$\times\prod_{f>\phi}M^{(j_f-j_{\pi(f)})D_f} \tag{2.26}$$

and

$$|Val(G^J)|'\leqslant(4K_0)^{|L(G)|}\prod_{v\in V_2(G)}(|\theta_v|_0\, M^{-j_{\pi(v)}})$$
$$\times\prod_{v\in V_4(G)}|\mathscr{U}_v|_0\, S_{\text{int}}\, S_{f,\text{ext}}\, S_{v,\text{ext}} \tag{2.27}$$

where $|\cdot|'$ is defined in (1.46),

$$S_{\text{int}}=\prod_{f>\phi,\ \text{internal}} M^{D_f(j_f-j_{\pi(f)})} \tag{2.28}$$

where the product is only over those forks of $t(G^J)$ such that G_f^J does not contain any external vertices, and

$$S_{f,\text{ext}}=\prod_{f>\phi,\ \text{external}} M^{\Delta_f(j_f-j_{\pi(f)})} \tag{2.29}$$

where the product is over those forks f of $t(G^J)$ such that G_f^J contains an external vertex of G, and

$$S_{v,\mathrm{ext}} = \prod_{v,\ \mathrm{external}} M^{\Delta_v(0 - j_{\pi(v)})} \tag{2.30}$$

where the product is over those vertices of G to which an external leg is joined. $\pi(v)$ is the highest fork such that G_f contains v and $\pi(f)$ is the predecessor fork of f, that is, the fork of $t(G^J)$ immediately below f.

Proof. See ref. 3. An improvement of (2.26) will be shown in Section 2.6. ∎

Remark 2.5. By definition,

$$D_f = \left(2\,|V_4(G_f)| + |V_2(G_f)| - \frac{E_f}{2}\right) - 2(|V_4(G_f)| + |V_2(G_f)| - 1)$$

$$= \frac{1}{2}(4 - E_f) - |V_2(G_f)| \tag{2.31}$$

If the graph G has no two-legged vertices, and if no *internal* subgraph G_f^J (i.e., G_f^J contains no external vertices) has $E_f = 2$, then $\Delta_v \leqslant -1/2$ and $\Delta_f \leqslant -1/2$, and

$$D_f = \tfrac{1}{2}(4 - E_f) \leqslant 0 \tag{2.32}$$

for all internal forks, so the scale sum $\sum_J |Val(G^J)|'$, where J runs over all labelings of G compatible with a fixed tree t,[2,3] will be finite. This is the rigorous counterpart of the remark in the Introduction that only insertions of two-legged diagrams give rise to divergences. Renormalization will be done by subtracting the value of the two-legged subgraphs on the Fermi surface. For this we need to introduce a projection onto the Fermi surface.

2.2. Localization Operator

The localization operator implements the projection onto the Fermi surface for functions defined on $\mathbb{R} \times \mathscr{B}$, and it is used to define the subtractions needed for renormalization. This projection can be defined in various ways, and so the localization operator is not uniquely determined. In the spherically symmetric case, there is exactly one choice that is rotationally invariant. Moreover, it does not matter which projection is chosen because rotational invariance implies that the value of any two-legged diagram $T(p_0, \mathbf{p})$ depends only on p_0 and $|\mathbf{p}|$. In the case without spherical symmetry, there is no such independence and hence no canonical choice of the projection, although the geometrically most natural one seems to be that which projects along integral curves of ∇e. We project $\mathbf{p}$ onto S differently, by moving it along the integral curve of the fixed vector field u transversal to S (see Lemma 2.1). This yields bounds in better norms than using ∇e (see Remark 2.2) because $\nabla e \in C^{k-1}$, but u may be chosen in C^{∞}.

Definition 2.6. Let δ be as in Lemma 2.1 and let $\chi \in C^{\infty}(\mathscr{B}, [0, 1])$ obey $\chi(x) = 1$ for $x \in U_{\delta}(S)$ and $\chi(x) = 0$ for $x \notin U_{2\delta}(S)$. Let $\mathbf{P}$ be as in Lemma 2.1(iii). For functions T: $\mathbb{R} \times \mathscr{B} \to X$, X any linear space, define

$$(\ell T)(q_0, \mathbf{q}) = \begin{cases} T(0, \mathbf{P}(\mathbf{q}))\,\chi(\mathbf{q}) & \text{if} \quad \mathbf{q} \in U_{2\delta}(S) \\ 0 & \text{otherwise} \end{cases} \tag{2.33}$$

If $T \in C^p(\mathbb{R} \times \mathscr{B}, X)$, then $\ell T \in C^q(\mathscr{B}, X)$, where $q = \min\{k, p\}$.

Lemma 2.7. Let $\mathscr{D}_u = u \cdot \nabla$ be the Lie derivative with respect to u, and T be differentiable on $\mathbb{R} \times U_{\delta}(S)$ with a bounded derivative. In terms of the coordinates (ρ, ω) introduced in Lemma 2.1,

$$(\ell T)(q_0, \mathbf{q}(\rho, \omega)) = T(0, \mathbf{q}(0, \omega)) \tag{2.34}$$

$$\frac{\partial}{\partial \rho} T(q_0, \mathbf{q}(\rho, \omega) = \left(\frac{\mathscr{D}_u T}{\mathscr{D}_u e}\right)(q_0, \mathbf{q}(\rho, \omega)) \tag{2.35}$$

In particular, $\mathscr{D}_u \mathbf{P} = 0$ and

$$\mathscr{D}_u \ell|_{U_{\delta}(S)} = 0 \tag{2.36}$$

For all $q = (q_0, \mathbf{q}) \in \mathbb{R} \times U_{\delta}(S)$,

$$|(1 - \ell)\, T(q)| \leqslant \frac{\sqrt{2}}{u_0}\, |iq_0 - e(\mathbf{q})|\, \max\{|\partial_0 T|_0, |\nabla T|_0\} \tag{2.37}$$

Proof. By the chain rule

$$\begin{aligned} \frac{\partial}{\partial \rho} T(q_0, \mathbf{q}(\rho, \omega)) &= \nabla T(q_0, \mathbf{q}(\rho, \omega)) \cdot \frac{\partial \mathbf{q}}{\partial \rho}(\rho, \omega) \\ &= \nabla T(q_0, \mathbf{q}(\rho, \omega)) \cdot \frac{\partial}{\partial \rho} \gamma_{\omega}(\tau(\rho)) \\ &= \nabla T(q_0, \mathbf{q}(\rho, \omega)) \cdot u(\mathbf{q}(\rho, \omega))\, \frac{\partial \tau}{\partial \rho}(\rho) \end{aligned}$$

So (2.35) follows from (2.11). $\mathscr{D}_u \mathbf{P} = 0$ then follows immediately from (2.35) with $T(q_0, \mathbf{q}) = \mathbf{P}(\mathbf{q})$. In other words, because the projection $\mathbf{P}(\mathbf{q}(\rho, \omega)) = \omega$

is constant along the integral curves of u, and the Lie derivative $\mathscr{D}_u$ is a directional derivative tangent to these integral curves, we have $\mathscr{D}_u \mathbf{P} = 0$. If $\mathbf{q} \in U_\delta(S)$, $\chi(\mathbf{q}) = 1$, so

$$(\mathscr{D}_u \ell T)(q_0, \mathbf{q}) = (u \cdot \nabla)\, T(0, \mathbf{P}(\mathbf{q})) = (\mathscr{D}_u \mathbf{P} \cdot \nabla T)(0, P(\mathbf{q})) = 0 \quad (2.38)$$

and

$$\begin{aligned} |(1 - \ell)\, T(q)| &= |T(q_0, \mathbf{q}) - T(0, \mathbf{P}(\mathbf{q}))| \\ &\leqslant |q_0| \cdot |\partial_0 T|_0 + |\mathbf{q} - \mathbf{P}(\mathbf{q})| \cdot |\nabla T|_0 \end{aligned} \quad (2.39)$$

so (2.37) holds by Lemma 2.1(iii). ∎

To put the localization operation into contact with the flow of effective actions, we define its action on a linear subspace of the Grassmann algebra given by "connected" polynomials of even degree. To define this subspace, we introduce some notation. The fermions in our model carry an index $\xi = (\alpha, p_0, \mathbf{p})$, where $\alpha \in \{\uparrow, \downarrow\}$, $p_0 \in \mathbb{R}$, and $\mathbf{p} \in \mathscr{B}$ for infinite volume and zero temperature. For temperature $T > 0$, $p_0 \in (2\mathbb{Z} + 1)\, \pi T$. For a periodic box of side L, $\mathbf{p} \in \mathscr{B} \cap (2\pi/L)\, \mathbb{Z}^d$. For $\xi = (\alpha, p_0, \mathbf{p})$ we denote $\psi(\xi) := \psi_\alpha(p_0, \mathbf{p})$ and similarly for $\bar{\psi}$. We also write $X = \{\uparrow, \downarrow\} \times \mathbb{R} \times \mathscr{B}$ and

$$\int_X ds(\xi)\, F(\xi) := \sum_{\alpha \in \{\uparrow, \downarrow\}} \int_{\mathbb{R}} dp_0 \int_{\mathscr{B}} d^d\mathbf{p}\, F(\alpha, p_0, \mathbf{p})$$

(and their obvious variations for $T > 0$ or finite volume).

Definition 2.8. We say that $Q \in \mathscr{Q}_c^k$ iff $Q = (Q_{2m,r})_{m \geqslant 0, r \geqslant 1}$, where for all $r \geqslant 1$ and $m \geqslant 0$:

(i) $Q_{2m,r}\colon X^{2m-1} \times \{\uparrow, \downarrow\} \to \mathbb{C}$,

$$(\xi_1, \dots, \xi_{2m-1}, \alpha_{2m}) \mapsto Q_{2m,r}(\xi_1, \dots, \xi_{2m-1}, \alpha_{2m})$$

is C^k and all derivatives up to order k are bounded uniformly on $X^{2m-1} \times \{\uparrow, \downarrow\}$.

(ii) For all $r \geqslant 1$, there is $\bar{m}(r) \geqslant 0$ such that for all $m \geqslant \bar{m}(r)$, $Q_{2m,r} = 0$.

(iii) $Q_{2m,r}$ is antisymmetric under permutations of momenta and spins, i.e., $\mathscr{A} Q_{2m,r} = Q_{2m,r}$, where $\mathscr{A}$ is the following operation. Define $p_{2m} = ((p_{2m})_0, \mathbf{p}_{2m}) \in \mathbb{R} \times \mathscr{B}$ by

$$p_{2m} = \sum_{i=1}^{m-1} (p_i - p_{m+i}) + p_m \quad (2.40)$$

and define $\xi_{2m} = (\alpha_{2m}, p_{2m})$. Then

$$\begin{aligned}
&\mathscr{A}Q(\xi_1,..., \xi_{2m-1}, \alpha_m) \\
&\quad = \frac{1}{(m!)^2} \sum_{\pi, \sigma \in \mathrm{Perm}(m)} \mathrm{sign}(\pi\sigma) \\
&\qquad \times Q(\xi_{\sigma(1)},..., \xi_{\sigma(m)}, \xi_{m+\pi(1)},..., \xi_{m+\pi(m-1)}, \alpha_{m+\pi(m)})
\end{aligned} \tag{2.41}$$

(iv) The polynomial in the Grassmann algebra associated to $Q \in \mathscr{Q}_c^k$ is the formal power series in λ

$$\begin{aligned}
Q(\psi, \bar{\psi}) = \sum_{r=1}^{\infty} \lambda^r \sum_{m=0}^{\bar{m}(r)-1} \int_{X^{2m}} \prod_{i=1}^{2m} ds(\xi_i)\, \delta^{\#}\left(\sum_{i=1}^{m} (p_i - p_{i+m})\right) \\
\times Q_{2m,r}(\xi_1 ..., \xi_{2m-1}, \alpha_m) \left(\prod_{i=1}^{m} \bar{\psi}(\xi_{m+i})\, \psi(\xi_i)\right)
\end{aligned} \tag{2.42}$$

Every fixed order in λ is a polynomial in the Grassmann variables. For convenience of notation, we somtimes write $Q^{(r)}_{\alpha_1 ... \alpha_m, \alpha_{m+1} ..., \alpha_{2m}}(p_1,..., p_{2m-1})$ for $Q_{2m,r}(\xi_1 ..., \xi_{2m-1}, \alpha_m)$. In this notation, the quadratic $(m=1)$ term in $Q(\psi, \bar{\psi})$ is given by the formal power series

$$\sum_{r=1}^{\infty} \lambda^r \sum_{\alpha_1, \alpha_2} \int d^{d+1}p\, \bar{\psi}_{\alpha_1}(p)\, Q^{(r)}_{\alpha_1, \alpha_2}(p)\, \psi_{\alpha_2}(p)$$

Definition 2.9. The localization operator $\ell\colon \mathscr{Q}_c^k \to \mathscr{Q}_c^k$ is defined as follows. For $Q \in \mathscr{Q}_c^k$ and all $r \geqslant 1$

$$\begin{aligned}
(\ell Q)_{2m,r} &= 0 \qquad \text{if} \quad m \geqslant 2 \\
(\ell Q)_{2,r}((\alpha_1, p_1), \alpha_2) &= Q_{2,r}((\alpha_1, (0, \mathbf{P}(\mathbf{p}_1)), \alpha_2) \\
(\ell Q)_{0,r} &= Q_{0,r}
\end{aligned} \tag{2.43}$$

In other words, for $Q(\psi, \bar{\psi})$ given by (2.42),

$$(\ell Q)(\psi, \bar{\psi}) = \sum_{r=1}^{\infty} \lambda^r \left(Q_{0,r} + \int d^{d+1}p\, \bar{\psi}_{\alpha_1}(p)\, Q^{(r)}_{\alpha_1 \alpha_2}(0, \mathbf{P}(\mathbf{p}))\, \psi_{\alpha_2}(p)\right)$$

2.3. Flow of Effective Actions

We review briefly the definition of effective actions and their flow, as given, e.g., in ref. 3. We introduce a cutoff that regulates the fermion propagator by restricting its support away from the Fermi surface, so that the formally divergent integrals discussed above are convergent as long as the cutoff is present. This can also be done in finite volume and the infrared cutoff can be removed before taking the volume to infinity. The flow is used to study the dependence of the Green functions on the cutoff as the latter varies. The propagator is decomposed linearly into a sum of slice propagators that are supported in thin shells around the Fermi surface. Because the decomposition is linear, the flow has a semigroup structure that allows one to view the Green functions as effective interactions where the fields with momenta that are away from the Fermi surface by an amount given by the cutoff are integrated out. Let $I \in \mathbb{Z}$, $I < 0$, be the infrared cutoff and decompose the cutoff propagator

$$C = \sum_{-1 \geqslant j \geqslant I} C_j \tag{2.44}$$

Define $\mathscr{G}_I^{\mathscr{V}}$ by

$$e^{\mathscr{G}_I^{\mathscr{V}}(\chi, \bar{\chi})} = \frac{1}{Z_I} \int d\mu_C(\psi, \bar{\psi})\, e^{\mathscr{V}(\psi + \chi, \bar{\psi} + \bar{\chi})} \tag{2.45}$$

where $d\mu_C$ denotes the Gaussian measure, i.e., the linear functional on the Grassmann algebra generated by the ψ and $\bar{\psi}$ defined to vanish for odd monomials and determined by its values for even monomials, which are

$$\int d\mu_C(\psi, \bar{\psi}) \prod_{i=1}^{n} \psi_{\alpha_i}(x_i)\, \bar{\psi}_{\beta_i}(y_i) = \det(C_{\alpha_i \beta_j}(x_i, y_j))_{1 \leqslant i, j \leqslant n} \tag{2.46}$$

In our case $C_{\alpha\beta}(x, y) = \delta_{\alpha\beta} \check{C}(x - y)$, so, using the Fourier expansion (1.18), we get in momentum space (in the sense of distributions)

$$\int d\mu_C(\psi, \bar{\psi}) \prod_{i=1}^{n} \psi_{\alpha_i}(p_i)\, \bar{\psi}_{\bar{\alpha}_i}(\bar{p}_i) = \det(\delta_{\alpha_i \bar{\alpha}_j}\, \delta(p_i - \bar{p}_j)\, C(p_i))_{ij} \tag{2.47}$$

where

$$C(p) = \sum_{-1 \geqslant j \geqslant I} C_j(p_0, e(\mathbf{p})) \tag{2.48}$$

$\mathscr{G}_I^{\mathscr{V}}$ is the generating functional for connected, amputated Green functions with infrared cutoff I and vertices given by $\mathscr{V}$, because, formally, a shift in the integration variables,

$$\mathscr{G}_I^{\mathscr{V}}(\chi, \bar{\chi}) = -(\bar{\chi}, C^{-1}\chi) + \log \frac{1}{Z_I} \int d\mu_C(\psi, \bar{\psi}) \times \exp[-(\bar{\psi}, C^{-1}\chi) - (\bar{\chi}, C^{-1}\psi) + \mathscr{V}(\psi, \bar{\psi})] \qquad (2.49)$$

indicates that $C^{-1}\chi$ and $\bar{\chi}C^{-1}$ appear as source terms. The effect of the C^{-1} is that propagators associated to external lines are removed. This is, by definition, the procedure to get Green functions that are amputated by the free propagator.

The unrenormalized expansion has $\mathscr{V}$ being the bare interaction. For the renormalized expansion we will allow $\mathscr{V}$ to depend on I because the counterterms will be I dependent. The factor

$$Z_I = \int d\mu_C(\psi, \bar{\psi})\, e^{\mathscr{V}(\psi, \bar{\psi})} \qquad (2.50)$$

ensures that $\mathscr{G}_I^{\mathscr{V}}(0, 0) = 0$. We now define precisely the fluctuation integrals used for the flow.

Definition 2.10. (i) Let $\mathscr{U} \in \mathscr{Q}_c^k$ and the covariance C be a bounded integrable C^k function on $\mathbb{R} \times \mathscr{B}$. Define

$$\mathscr{R}(C, \mathscr{U})(\chi, \bar{\chi}) = \log \frac{1}{\zeta(C, \mathscr{U})} \int d\mu_C(\psi, \bar{\psi})\, e^{\mathscr{U}(\psi + \chi, \bar{\psi} + \bar{\chi})} \qquad (2.51)$$

where $\zeta(C, \mathscr{U}) = \int d\mu_C(\psi, \bar{\psi})\, e^{\mathscr{U}(\psi, \bar{\psi})}$ so that $\mathscr{R}(C, \mathscr{U})(0, 0) = 0$. Also, define

$$\mathscr{E}(C, \mathscr{U})(\chi, \bar{\chi}) = \mathscr{R}(C, \mathscr{U})(\chi, \bar{\chi}) - (\mathscr{U}(\chi, \bar{\chi}) - \mathscr{U}(0, 0)) \qquad (2.52)$$

(ii) Let G be a connected graph with n vertices $v_1, \ldots, v_n$ and $2m$ external legs such that every vertex v_i has m_i ingoing and m_i outgoing legs (incidence number $2m_i$), and let

$$\begin{aligned} \mathscr{U}_{v_i}\colon\ & X^{2m_i - 1} \times \{\uparrow, \downarrow\} \to \mathbb{C} \\ & (\xi_1 \ldots, \xi_{2m_i - 1}, \alpha_{2m_i}) \mapsto \mathscr{U}_{v_i}(\xi_1, \ldots, \xi_{2m_i - 1}, \alpha_{2m_i}) \end{aligned} \qquad (2.53)$$

satisfy (i) and (iii) of Definition 2.8. Let $J\colon L(G) \to \{z \in \mathbb{Z}\colon z < 0\}$ be a labeling of G. The value of G^J is defined as the function $Val(G^J)(C, \mathscr{U}_{v_1}, \ldots, \mathscr{U}_{v_n})$: $X^{2m-1} \times \{\uparrow, \downarrow\} \to \mathbb{C}$, determined by

$$\delta^{\#}\left(\sum_{i=1}^{m}(q_i-q_{m+i})\right) Val(G^J)(C,\mathscr{U}_{v_1},\dots,\mathscr{U}_{v_n})(\eta_1,\dots,\eta_{2m-1},\beta_{2m})$$
$$=\sum_{\text{spins}}\int\prod_{l\in L(G)}(C_{j_l}((p_l)_0,e(\mathbf{p}_l))_{\alpha_l,\bar\alpha_l}\,d^{d+1}p_l)$$
$$\times\prod_{i=1}^{n}\delta^{\#}\left(\sum_{k=1}^{m_i}(p_k^{(i)}-p_{m+k}^{(i)})\right)$$
$$\times\mathscr{U}_{v_i}((p_1^{(i)},\alpha_1^{(i)}),\dots,(p_{2m_i-1}^{(i)},\alpha_{2m_i-1}^{(i)}),\alpha_{2m_i}^{(i)}) \tag{2.54}$$

where $\eta_i=(q_i,\beta_i)$, and $\sum_{\text{spins}}$ means that all $\alpha_k^{(i)}$ are summed over $\{\uparrow,\downarrow\}$. If the line l joins the outgoing leg k of vertex v_i to the incoming leg k' of $v_{i'}$, then $\alpha_l=\alpha_k^{(i)}$, $\bar\alpha_l=\alpha_{k'}^{(i')}$, and the momenta $p_k^{(i)}=p_{k'}^{(i')}=p_l$.

(iii) The set of all connected graphs with $2m$ external legs and with n vertices $v_1,\dots,v_n$ where v_i has m_i incoming and m_i outgoing legs (incidence number $2m_i$) is denoted by $Gr(n,m;m_1,\dots,m_n)$. When $n=1$, the graphs are required to have at least one internal particle line.

Lemma 2.11. (i) $Val(G^J)(C,\mathscr{U}_{v_1},\dots,\mathscr{U}_{v_n})$ is well defined and a C^k function of the external momenta.

(ii) $\mathscr{E}$ and $\mathscr{R}$ are well-defined formal power series in λ. They map $\mathscr{Q}_c^k$ to $\mathscr{Q}_{c'}^k$.

(iii) Expanding in powers of $\mathscr{U}$,

$$\mathscr{E}(C_j,\mathscr{U})=\sum_{n=1}^{\infty}\frac{1}{n!}\mathscr{E}^{(n)}(C_j,(\mathscr{U},\mathscr{U},\dots,\mathscr{U})) \tag{2.55}$$

($\mathscr{U}$ appears n times), and expanding the $\mathscr{E}^{(n)}$ in λ and the fields as well, $\mathscr{E}$ has the expansion [see also Definition 2.8, (iv)]

$$\mathscr{E}(C_j,\mathscr{U})(\chi,\bar\chi)=\sum_{r=1}^{\infty}\lambda^r\sum_{m=1}^{\bar m(r)}\int\prod_{i=1}^{2m}ds(\eta_i)$$
$$\times\delta^{\#}\left(\sum_{i=1}^{n}(q_i-q_{m+i})\right)\prod_{k=1}^{m}(\bar\chi(\eta_k)\,\chi(\eta_{k+m}))$$
$$\times E_{j,2m,r}(\mathscr{U})(\eta_1,\dots,\eta_{2m-1},\beta_{2m}) \tag{2.56}$$

where $\eta_k=(q_k,\beta_k)$, and the kernels $E_{j,2m,r}$ are the following sum of values of Feynman diagrams:

$$E_{j,2m,r}(\mathscr{U})=\sum_{n=1}^{\infty}\frac{1}{n!}E_{j,2m,r}^{(n)}(\mathscr{U},\dots,\mathscr{U}) \tag{2.57}$$

with

$$E^{(n)}_{j,2m,r}(\mathcal{U}_{v_1},\ldots,\mathcal{U}_{v_n})(\eta_1,\ldots,\beta_{2m})$$

$$= \sum_{\substack{r_1,\ldots,r_n \geq 1 \\ r_1+\cdots+r_n=r}} \; \sum_{m_1,\ldots,m_n\geq 1} \; \sum_{G\in Gr(n,m;m_1,\ldots,m_n)}$$

$$\times \operatorname{sign}(G)\, \mathcal{A}\, Val(G^{(j)})(C,\mathcal{U}_{v_1,r_1},\ldots,\mathcal{U}_{v_n,r_n})(\eta_1,\ldots,\beta_{2m}) \quad (2.58)$$

Here $\mathcal{A}$ denotes the antisymmetrization operator defined in (2.41), $\operatorname{sign}(G)\in\{1,-1\}$ is a sign factor determined by the structure of G, (j) denotes the labeling $j_l=j$ for all l, and $\mathcal{U}_{v_k,r_k}$ denotes the coefficient of order r_k in the formal expansion of $\mathcal{U}_{v_k}$ in powers of λ. The sum over the number of effective interaction vertices n in (2.57) is a finite sum with $n\leqslant r$. The sums in (2.58) are finite because the interaction $\mathcal{U}\in\mathcal{Q}^k_c$ and in particular satisfies (ii) of Definition 2.8.

Remark 2.12. Although lengthy, (2.58) is easy to interpret: At every scale, the Green function is expanded in a formal power series in λ. In every order in λ, the functional is expanded in powers of the external (unintegrated) fields χ and $\bar{\chi}$. The term with m factors of χ and m factors of $\bar{\chi}$ contributes to the $2m$-point function, and is given by the sum over all connected Feynman diagrams with $2m$ external legs, built from the effective vertices $\mathcal{U}$.

Proof. The momentum conservation delta functions at every vertex can provide a set of loop momenta using some choice of a spanning tree for G in the standard way. Since G is connected, only the global momentum conservation delta function remains. It is then obvious by the properties of the integrand to see that the function that multiplies this delta function is C^k and so (i) follows. Since (ii) follows from (iii) and (i), it suffices to show (iii). Although this is quite standard, we briefly describe how the expansion in Feynman graphs comes about, since it is also very simple. By definition, $\mathcal{E}^{(n)}$ takes out the terms proportional to $\mathcal{U}_{v_1}\ldots\mathcal{U}_{v_n}$, so it is obviously linear in every $\mathcal{U}_{v_k}$. Inserting the expansion for every $\mathcal{U}\in\mathcal{Q}^k_c$, we obtain the sum over the r_i and m_i. Note that by definition of $\mathcal{Q}^k_c$, all sums over r_i start at 1 and therefore $\mathcal{E}$ and $\mathcal{R}$ contain no zeroth-order terms in λ. Furthermore, $m_i\leqslant\bar{m}_i(r_i)$, so all sums contain only finitely many terms. The rules for Gaussian integration (2.47) then join outgoing ($\bar{\psi}$) legs of $\mathcal{U}_v$ to ingoing (ψ) legs of $\mathcal{U}_{v'}$. The result can be translated into a sum over Feynman graphs by joining v and v' by a line and using the definition of the value of a graph given above. Since the logarithm is taken, only connected graphs contribute (see, e.g., ref. 9). Consequently

$$\bar{m}(r) \leqslant \max_{\substack{v, r_1, \ldots, r_n \geqslant 1 \\ r_1 + \cdots + r_n = r}} \sum_{i=1}^{n} [\bar{m}_i(r_i) - 1] \quad \blacksquare$$

Remark 2.13. In fact, because of the fermionic nature of the fields, defining suitable norms on $\mathscr{Q}_c^k$, analyticity holds in a disk $\{|\lambda| < \lambda_0\}$, where λ_0 depends on the cutoff I. Since we consider the formal expansion only, we do not need to make use of that here.

The flow is now obtained by successively integrating out the momenta of shells around the Fermi surface. Since C is a sum of covariances, the Gaussian measure factorizes into a product $\prod_{j=I}^{-1} d\mu_{C_j}$, and $\mathscr{G}_I^{\mathscr{V}}$ can be written as the endpoint of the sequence

$$\mathscr{G}_j^{\mathscr{V}} = \mathscr{R}\left(\sum_{i=j}^{-1} C_i, \mathscr{V}\right) \tag{2.59}$$

The sequence starts with $\mathscr{G}_0^{\mathscr{V}} = \mathscr{V}$ and may be obtained by iteration of

$$\mathscr{G}_j^{\mathscr{V}} = \mathscr{R}(C_j, \mathscr{G}_{j+1}^{\mathscr{V}}) = \mathscr{G}_{j+1}^{\mathscr{V}} + \mathscr{E}(C_j, \mathscr{G}_{j+1}^{\mathscr{V}}) \tag{2.60}$$

The recursion can be summed to get, assuming $\mathscr{V}(0, 0) = 0$.

$$\mathscr{G}_j^{\mathscr{V}}(\chi, \bar{\chi}) = \mathscr{V}(\chi, \bar{\chi}) + \sum_{-1 \geqslant i \geqslant j} \mathscr{E}(C_i, \mathscr{G}_{i+1}^{\mathscr{V}})(\chi, \bar{\chi}) \tag{2.61}$$

Lemma 2.11 implies the following result.

Lemma 2.14. Let $e \in C^k(\mathscr{B}, \mathbb{R})$, $k \geqslant 1$, and assume A2. If the initial interaction $\mathscr{V} \in \mathscr{Q}_c^k$, then for any scale j, $\mathscr{G}_j^{\mathscr{V}} \in \mathscr{Q}_c^k$.

Taking the initial interaction to be the bare one, $\mathscr{V} = \lambda V$, yields the sequence of unrenormalized effective actions which diverges as $I \to -\infty$ for the reasons discussed in the Introduction.

The renormalized Green functions are constructed by modifying the interaction such that the Fermi surface of the interacting system, that is, the singular surface of the interacting fermion propagator, stays fixed. This requires a specific choice of $\mathscr{V}$ which we denote as $\mathscr{G}_0^I$, the I indicating the dependence on the infrared cutoff. Using the similar notation

$$\mathscr{G}_j^{\mathscr{G}_0^I} = \mathscr{G}_j^I$$

for the $\mathscr{G}_j$ obtained from this interaction by (2.59), we require, as a condition on $\mathscr{G}_0^I$,

$$\ell \mathscr{G}_I^I = \ell \mathscr{G}_0^I + \sum_{I \leqslant i \leqslant -1} \ell \mathscr{E}(C_i, \mathscr{G}_{i+1}^I) = 0 \tag{2.62}$$

so

$$\ell \mathscr{G}_0^I = - \sum_{I \leqslant i \leqslant -1} \ell \mathscr{E}(C_i, \mathscr{G}_{i+1}^I) \tag{2.63}$$

Since all $\mathscr{G}_j^I$ are functionals of $\mathscr{G}_0^I$, this is not a definition but an equation to the solved by $\mathscr{G}_0^I$. There are further conditions on $\mathscr{G}_0^I$: We want the form of the interaction to be similar to the original one. Only terms bilinear in the fermion fields shall be generated:

$$(1-\ell)\, \mathscr{G}_0^I = (1-\ell)\, \lambda V = \lambda V \tag{2.64}$$

(2.63) can be solved order by order in λ, that is, as a formal power series in λ,

$$\mathscr{G}_0^I = \sum_{r=1}^{\infty} \mathscr{G}_{0,r}^I \lambda^r \tag{2.65}$$

as follows. All $\mathscr{G}_j^I$ are formal power series in λ, with no zeroth-order term, since they are connected Green functions (and since the free part is subtracted from the two-point function). One proceeds inductively in r, the order in λ, in (2.63). To get the left side in order r, only counterterms in $\mathscr{G}_{i+1}^I$ up to order $r-1$ are needed on the right side of the equation. No graph contributing to the right-hand side of (2.63) can consist of a single two-legged vertex with no internal lines. The left side of (2.63) can simply be used to give a recursive definition for the counterterms.

Definition 2.15. The generating functional for the renormalized Green functions is obtained by the flow (2.61) with initial interaction $\mathscr{G}_0^I = \lambda V + \mathscr{K}^I$, where V is the interaction given by (1.23) and the counterterms $\mathscr{K}^I$ are defined as a formal power series in λ by

$$\mathscr{K}^I(\chi, \bar{\chi}) = -\ell \sum_{I \leqslant i \leqslant -1} \mathscr{E}(C_i, \mathscr{G}_{i+1}^I)(\chi, \bar{\chi}) \tag{2.66}$$

and we shall call their formal power series expansion in terms of λ the renormalized perturbation expansion. The expansion coefficients of $\mathscr{G}^I$ given by Definition 2.8(iv) are the renormalized, amputated, connected

Green functions. More explicitly, the rth-order $2m$-point function on scale $j \geqslant I$, $G^I_{j,2m,r}$, is obtained by replacing the $\mathscr{E}(C_j, \mathscr{U})$ by

$$\mathscr{G}^I_j = \mathscr{G}^{\mathscr{G}^I_0}_j$$

and the $E_{j,2m,r}$ by $G^I_{j,2m,r}$ in (2.56) and the functions $G^I_{2m,r}$ from Section 1.5 are defined as

$$G^I_{2m,r} = G^I_{I,2m,r} \tag{2.67}$$

The counterterms are of the form

$$\mathscr{K}^I(\chi, \bar{\chi}) = \sum_\alpha \int_{\mathbb{R}\times\mathscr{B}} d^{d+1}p\, \bar{\chi}_\alpha(p)\, K^I(\mathbf{p})\, \chi_\alpha(p) \tag{2.68}$$

where K^I is a formal power series in λ,

$$K^I(\mathbf{p}) = \sum_{r=1}^{\infty} \lambda^r K^I_r(\mathbf{p}) \tag{2.69}$$

The $G^I_{j,2m,r}$ are all of order $r \geqslant 1$ in the coupling λ. In particular, the two-point function has the zeroth-order propagator subtracted. Hence the formula (1.47) for the self-energy. The recursion formula (2.60) can be written for the kernels G^I as

$$G^I_{j,2m,r} - G^I_{j+1,2m,r} = E_{j,m,r}(\mathscr{G}^I_{j+1}) \tag{2.70}$$

To show convergence of the renormalized Green functions in the limit as the cutoff I is removed, $I \to -\infty$, it is convenient to arrange (2.61) in the form

$$\mathscr{G}^I_j = \lambda V + \sum_{i \geqslant j} (1 - \ell)\, \mathscr{E}(C_i, \mathscr{G}^I_{i+1}) + \sum_{i<j} (-\ell)\, \mathscr{E}(C_i, \mathscr{G}^I_{i+1}) \tag{2.71}$$

Iteration of this equation for $\mathscr{G}^I_j$ generates a tree structure, corresponding to layers of $\mathscr{G}_j$. Expanding this out to scale zero, one recovers the scaled graph G^J from Lemma 2.4. For the unrenormalized expansion, the scales of lines in $G_{f'}$ are strictly higher than those in G_f if $f' > f$ on the tree. In case of the renormalized expansion, this holds for r-forks, generated by the second term in (2.71). The third term in (2.71) gives to the c-forks of the tree. The scales of a c-fork f are summed from I to $j_{\pi(f)}$ since $i \leqslant j$ in the third term of (2.71).

The semigroup structure of the renormalization flow is obvious from the way it is defined by fluctuation integrals. It is a consequence of the linear decomposition of the covariance C into a sum of C_j's. It allows one to interpret the formula for $\mathscr{G}_j$ in various ways. By definition, $\mathscr{G}_j$ is the amputated connected Green function with infrared cutoff j. Alternatively, one can also view $\mathscr{G}_{j+1}$ as an effective action, i.e., the $G^I_{j+1,2m}$ are vertex functions of effective interaction vertices with $2m$ external legs. These vertices are connected by $C_{j'}$-lines to form the effective action on scale j, $\mathscr{G}_j$. The process of expanding different parts of the tree, or equivalently, expanding the effective vertices in terms of higher scale objects, can be done to various degrees. One can choose to iterate selected parts of the tree, i.e., resolve selected vertices up to a certain higher scale or "trim the tree" at a fork f by regarding the subgraph G^J_f as a vertex with $E(G^J_f)$ external lines and vertex factor $\delta^{\#}(p_{f,\text{out}} - p_{f,\text{in}})\, Val(G^J_f)$. We shall make use of three variations on this theme, which we now briefly describe.

Remark 2.16. (i) Resolve every vertex up to scale zero, as described above; this gives sums over values of the standard labeled graphs G^J of Lemma 2.4. More precisely, this leads to the following formula for the amputated connected Green functions:

$$G^I_{2m,r} = \sum_{j \geqslant I} \sum_{t} \prod_{f \in t} \frac{1}{n_f!} \sum_{G} \sum_{J \in \mathscr{J}(t,j)} Val(G^J) \tag{2.72}$$

where [as follows from (2.56), (2.57); see also Section VI of ref. 2] the second sum is over all planar trees t with r leaves. The root is denoted ϕ, and for each fork f, $n_f \geqslant 1$ is the number of upward branches. ($n_f = 1$ is possible because we do not use normal ordering). The factorial is that from (2.56). The sum over graphs G runs over all G compatible with t, that is, connected graphs with $2m$ external legs, r ordered vertices, constructed according to the Feynman rules of the model. The leaves of t correspond to the four-legged interaction vertices of G. For any fork $f \in t$, there is a connected subgraph G_f of G, such that the quotient graph $\tilde{G}(\{f\})$ [obtained by replacing all G_g with $\pi(g) = f$ by effective vertices] has n_f vertices. The set $\mathscr{J}(t, j)$ of scale families J consists of all $(j_f)_{f \in t}$ ordered according to the partial ordering given by the tree t,

$$\begin{aligned} \mathscr{J}(t, j) = \{ (j_f)_{f \in t} \colon j_\phi = j;\ & \text{if } f \in t \text{ is not a c-fork, } j_f \in \{ j_{\pi(f)} + 1, \dots, 0\}; \\ & \text{if } f \text{ is a c-fork, } j_f \in \{ I, \dots, j_{\pi(f)} \} \} \end{aligned} \tag{2.73}$$

This definition is understood recursively, i.e., the root scale j_ϕ is fixed to j; if f is a c-fork with $\pi(f) = \phi$, then j_f runs from I to j_ϕ. If $\pi(f) = \phi$, but f is not a c-fork, j_f runs from $j_\phi + 1$ to 0. This assignment of scales is now continued upward on the tree, determining the range of j_f in terms of $j_{\pi(f)}$

and the r/c label on the fork. All leaves b of the tree t have scale zero, and the vertices of G associated to them are the interaction vertices of the original action. The labeling of the graph G, $l \mapsto j_l$, is: all lines l in $\tilde{G}(\{f\})$ get scale $j_l = j_f$. Finally, $Val(G^J)$ is defined according to the Feynman rules for labeled graphs, with a propagator of scale j_l associated to each line l. In our case, there is no hard/soft labeling for the lines because we do not use normal ordering.

(ii) Resolve everything except for one-particle irreducible (1PI) two- and four-legged insertions. More algorithmically, let G be a graph contributing to $\mathcal{G}_j$. For every vertex v, $\mathcal{U}_v$ is again the sum of values of graphs on scale $\geqslant j+1$. If v has $\leqslant 4$ legs and is 1PI leave it. Otherwise repeat the same procedure for the graph whose value is $\mathcal{U}_v$. Continue to resolve until all graphs that are not resolved are 1PI (for details, see Section 2.7). The result is a labeled graph G' that has no nontrivial one-particle irreducible two- or four-legged subdiagrams, but instead two- and four-legged vertices with scale-dependent vertex functions. This will be used to trace back the factorials in values of individual graphs (the reason for their occurrence are the nonoverlapping four-legged subgraphs) and to order the inductive proofs, since the scale-dependent vertex functions are themselves values of subgraphs of lower order. The vertices are scale dependent because the trimming procedure splits the summation over $\mathcal{J}$. Trimming a tree t at a fork ψ decomposes t into two subtrees t_1 and t_2 with ψ as a leaf of t_1 and ψ the root of t_2. Then

$$\mathcal{J}(t, j) = \{(j_f)_{f \in t} : J_1 = (j_f)_{f \in t_1} \in \mathcal{J}_1 = \mathcal{J}(t_1, j) \text{ and } J_2 = (j_f)_{f \in t_2} \in \mathcal{J}_2 = \mathcal{J}(t_2, j_\psi)\} \tag{2.74}$$

The vertex function $\mathcal{P}_w V_w$ of the vertex w in G' that corresponds to G^J_ψ is one scale j_ψ, and it is obtained by summing over the scales in $\mathcal{J}_2$, keeping those in $\mathcal{J}_1$ fixed,

$$V_w = \sum_{J_2 \in \mathcal{J}_2} Val(G^{J_2}_\psi) \tag{2.75}$$

The projection $\mathcal{P}_w \in \{\ell, 1-\ell, 1\}$ is given by the r/c labeling of the forks of t.

(iii) Resolve according to families of nonoverlapping subtrees rooted at forks belonging to two-legged diagrams; this will play a major technical role in the estimates of the derivative with respect to the band structure e. For details, see Section 2.5.

For further reference, we give the formula for K_r^I explicitly,

$$K_r^I(\mathbf{p}) = -\sum_G \sum_{j=I}^{-1} \sum_{t \sim G} \prod_{f \in t} \frac{1}{n_f!} \sum_{J \in \mathcal{J}(t,j)} Val(G^J)(0, \mathbf{P}(\mathbf{p})) \tag{2.76}$$

Note that K_r^I actually only depends on $\mathbf{P}(\mathbf{p}) \in S$. The sum over G is over all two-legged and one-particle irreducible (1PI) graphs G with r interaction vertices. The graphs have to be 1PI since $e(\mathbf{P}(\mathbf{p}))=0$ implies $C_j(0,0)=0$, and since the value of a 1P-reducible two-legged graph would contain such a factor. Note also that $K_r^I(\mathbf{p}) \in \mathbb{R}$ for all $\mathbf{p}$ by **A1**. This follows from (2.66) and a change of integration variables $\psi(p_0, \mathbf{p}) \rightarrow \psi(-p_0, \mathbf{p})$ in the functional integral defining ε.

2.4. Nonoverlapping Graphs

In this section, we give an explicit characterization of two- and four-legged graphs that do not contain any overlapping loops. These graphs turn out to be dressed bubbles in the four-legged case and graphs of the type encountered in the Hartree–Fock resummation in the two-legged case.

To make contact with the graph structure in our problem, and for convenience of the reader, we shown explicitly how certain low-order diagrams look when the interaction lines are collapsed to four-fermion vertices; see Fig. 6. Graphs 1, 5, and 6 each contains two loops which do not overlap. The last three graphs each contains two loops that do overlap.

We wish to single out those graphs which have overlapping loops. Their value contains a volume integral that can be bounded by the function I_2, defined in (3.4), which gives an additional convergence factor in scale sums. This will serve to show that derivatives converge and that a large class of 4-forks is actually not marginal, that is, that the power counting behaviour $D_f = 0$ is not saturated. For the graphs without overlapping loops, there is no such improvement. But these graphs have a rather special structure (see Fig. 6). In particular, the momentum of the external line will not enter in any of the loop lines if the graph is two-legged and nonoverlapping. The two lemmas in this section characterize graphs G that have no overlapping loops explicitly for $E(G)=2$ or 4. They are stated for the more general class of graphs that arise naturally when expanding the fluctuation integral for the effective action at some scale [see (2.54) and below it].

In the following, let G be a connected graph constructed from particle lines and generalized vertices v that all have an even incidence number $E_v \geqslant 2$. Such graphs occur naturally in the flow of effective actions. We call G one-particle irreducible (1PI) if any internal particle line of G can be cut without disconnecting the graph. We also use $L(G)$ for the set of all internal lines of G, $E(G)$ for the set of external lines of G, $V(G)$ for the set of

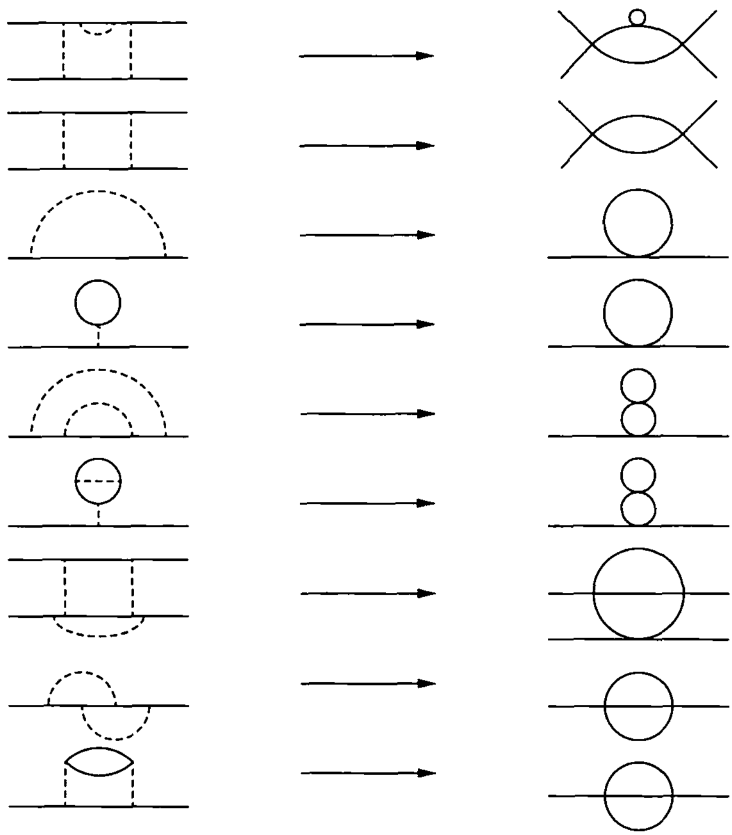

Fig. 6. Replacement of interaction lines by vertices.

all vertices, and $V_k(G)$ for the set of all vertices with incidence number k. For $v \in V(G)$, $G - v$ denotes the graph in which v and all lines going into v are deleted. For $l \in L(G)$, $G - l$ denotes the graph in which only the line l is removed (but not its endpoints). Denote the set of directed lines of G by $\mathscr{L}(G)$,

$$\mathscr{L}(G) = \{(l, v, w) \in L(G) \times V(G) \times V(G): l \text{ connects } v \text{ and } w\}$$

Definition 2.17. (i) Let $n_1, n_2 \in \mathbb{N}_0$, $n_1 \leqslant n_2$. A path P in G is a map P: $\{n_1,..., n_2) \to \mathscr{L}(G)$, $n \mapsto (l_n, v_n, w_n)$, such that for all $n \in \{n_1,..., n_2 - 1\}$, $w_n = v_{n+1}$, and such that each vertex of G is visited at most once by P.

(ii) A loop in G is a map P: $\{0,..., s\} \to \mathscr{L}(G)$ such that $P|_{\{0,...,s-1\}}$ is a path, and [in the notation of (i)], $w_s = v_0$, and the line from v_s to w_s

(a) (b)

Fig. 7. (a) A self-intersecting walk, (b) a self-contraction.

is a line of G (the case $s=0$ is a line from a vertex to itself, also called "self-contraction").

(iii) The trace $\theta(P)$ of the path or loop P is defined as the subgraph consisting of lines and vertices visited by P.

(iv) We say that two loops P_1 and P_2 are independent if their traces are distinct, $\theta(P_1) \neq \theta(P_2)$.

For example, under Definition 2.17, the object shown in Fig. 7a is not a path because it is self-intersecting. However, the object shown in Fig. 7b is a loop consisting of one line (a "self-contraction").

Remark 2.18. (i) We sometimes write the path as a finite sequence $(P(n_1),..., P(n_2))$.

(ii) If P is a path, so is its inversion P^{-1}, defined as going over the same lines as P, but in the opposite direction. If P is a loop, so is its shift by m, P_m, defined as $P_m(l) = P(l - m \bmod s)$.

(iii) Usually a loop is defined to be an element of the first homology group $H_1(G, \mathbb{Z})$. For the purposes of the following analysis of overlapping and nonoverlapping graphs, it does not really matter which of the definitions one takes.

(iv) Let T be a spanning tree for a graph G. Let $l \in L(G) \backslash L(T)$. Then the subgraph of G obtained by taking the union of l with the linear subtree of T that joins the endpoints of l is the trace of a loop under Definition 2.17.

Definition 2.19. G is called overlapping if there is a line of G which is part of two independent loops.

Remark 2.10. (i) If G is nonoverlapping and S a connected subgraph, then S is nonoverlapping.

(ii) If G is nonoverlapping, and $\tilde{G}$ a quotient of G obtained by replacing a connected subdiagram with a vertex, then $\tilde{G}$ is nonoverlapping.

(iii) If G is connected and S a subgraph that is overlapping, then G is overlapping.

(iv) If G is a nonoverlapping graph, and $\tilde{G}$ is obtained from G by forming a self-contraction of two external legs of a vertex v of G, then $\tilde{G}$ is nonoverlapping.

(v) If G contains a subgraph consisting of two vertices v_1 and v_2 joined by $n \geqslant 3$ lines $l_1,..., l_n$, then G is overlapping.

Proof. (i), (iii), and (iv) are obvious. (ii) Let $\tilde{G}$ be a quotient of G obtained by replacing a connected subdiagram H by a vertex. Let $\tilde{G}$ be overlapping. Then there are two independent loops L_1 and L_2 in $\tilde{G}$ that have a line $l \in \tilde{G}$ in common. As a path in G, L_1 either crosses at most one external vertex of H, in which case L_1 is still a loop in G, or it stops at two distinct external vertices of H. Since H is connected, there is a path connecting these vertices, and the composition of L_1 with this path is a new loop in G that still contains l. Similarly, L_2 either is already a loop in G or can be completed to one, and so G is also overlapping. This shows (ii). (v) Let the vertices be v_1 and v_2. Since $n \geqslant 3$, the loop L_1 going from v_1 to v_2 over l_1 and back over l_2 and the path L_2 going from v_1 to v_2 over l_1 and back over l_3 are independent. Both contain l_1. So the subgraph is overlapping, and the same follows for G itself by (iii). ▌

Definition 2.21. Let G be a connected graph with two external legs and N vertices all having even incidence number.

(i) If $G_1,..., G_n$ are two-legged graphs, the strings $G_1 \ldots G_n$ is the graph shown in Fig. 8a. The G_i may be two-legged vertices (i.e., vertices with incidence number two).

(ii) G is called a self-contracted two-legged (ST) diagram if G consists only of one two-legged vertex with two external legs or if G has exactly one vertex v_1 to which both external legs of G connect, all other vertices have two legs, and the remaining legs of v_1 are joined pairwise by strings of two-legged vertices to form loops. See Fig. 8b.

(iii) A generalized ST diagram (GST) with N vertices is defined recursively: if $N=1$, G is an ST diagram. If $N \geqslant 2$ and GST are defined for all $N' \leqslant N-1$, a GST with N vertices is a graph such that G has exactly one *external* vertex v_1 to which the two external legs of G join, and all other legs of v_1 are joined by strings of GST with at most $N-1$ vertices, to form loops (which we call "generalized self-contractions"). For examples, see Figs. 8c and 8d; in Fig. 8c the GST insertions are marked by crosses.

Lemma 2.22. Let G be a connected graph with two external legs and all vertices of G having an even incidence number. If G is nonoverlapping, it is a string of GST graphs. If G is nonoverlapping and 1PI, then G is a GST graph.

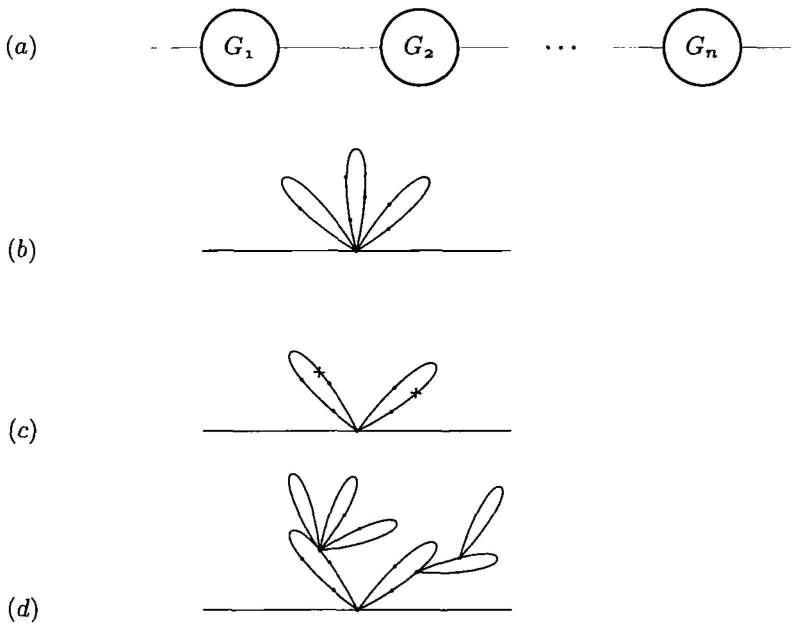

Fig. 8. (a) A string of two-legged diagrams, (b) an ST diagram, (c, d) GST diagrams.

Proof. The second statement is obvious, given the first. To prove the first, do induction in the number of vertices N. For $N=1$, an $E=2$ graph with one vertex must obviously be an ST diagram. It is instructive to look at $N=2$ first. There are two cases: (1) only v_1 has incident external legs, and (2) v_1 and v_2 both join to an external leg.

(1) Denote the incidence number of v_1 by n_1 and that of v_2 by n_2. Since two legs of v_1 are external and every self-contraction binds two legs, there must be an even number n of lines between v_1 and v_2 (see Fig. 9). If $n \geqslant 4$, G is overlapping by Remark 2.20(v) (there are $n-1 \geqslant 3$ independent loops containing any of the lines between v_1 and v_2). So $n=2$, which means that the graph is a GST.

(2) n_1 and n_2 are even, and v_1 and v_2 each bind one external leg of G. Since self-contractions bind an even number of lines, v_1 and v_2 must be joined by an odd number n of lines. If $n \geqslant 3$, G would be overlapping by Remark 2.20(v). So $n=1$, and G is a string of two ST graphs.

Let $N \geqslant 2$ and assume the lemma to the true for nonoverlapping graphs with N' vertices, $N' \leqslant N-1$, and the G be a nonoverlapping graph with N vertices and $E=2$.

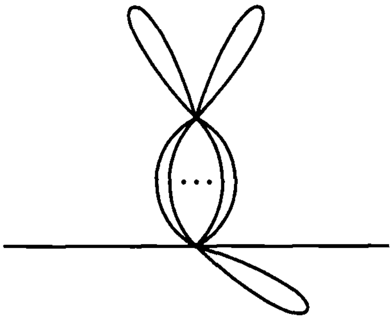

Fig. 9. An ST diagram with two vertices.

(1) If there is only one external vertex, v_1, G takes the form shown in Fig. 10a. Decomposing the subgraph $B = G - v_1$ into its connected components $C_1,..., C_l$, we see that G must be as drawn in Fig. 10b. Denote the number of lines joining v_1 and C_k by n_k. Let $k \in \{1,..., l\}$. Since all vertices in C_k have an even incidence number and legs are joined pairwise to form internal lines of C_k, the number n_k of external legs of C_k is even. As in the case $N=2$, if $n_k \geqslant 4$, the subgraph consisting of v_1 and C_k shown in Fig. 10c is overlapping by Remark 2.20(v), (ii). By Remark 2.20(iii), so is G. Therefore $n_k = 2$ for all $k \in \{1,..., l\}$ and, by Remark 2.20(i), being a subdiagram of the nonoverlapping graph G, C_k is nonoverlapping and two-legged with even-legged vertices and at most $N-1$ vertices. By the inductive hypothesis, C_k is a string of GST, so G is a GST by definition.

(2) If there are two external vertices v_1 and v_2, let $G_1 = G - v_2$ be the graph obtained from G by deleting v_2 and all the lines going into it. Let $C_1,..., C_r$ be the connected components of G_1, where C_1 contains v_1. Then G takes the form drawn in Fig. 11a. Consider the quotient graph G_2 where all C_k are replaced by vertices c_k (see Fig. 11b). Denote the number of lines from v_2 to c_k by n_k. Then for all $k \geqslant 2$, n_k must be even, since all the

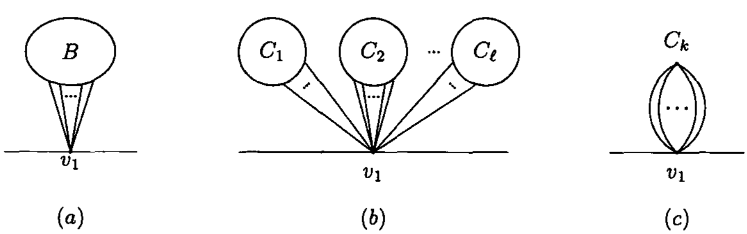

Fig. 10. The case of one external vertex.

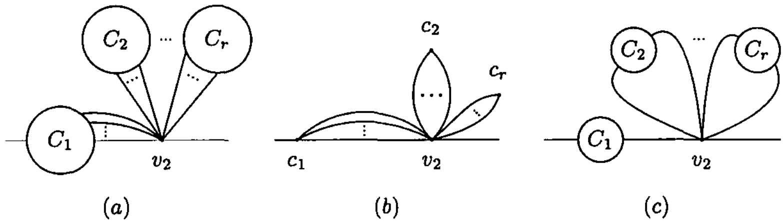

Fig. 11. The case of two external vertices.

vertices of the graph C_k have even incidence number. Since both c_1 and v_2 join to one external leg, the number n_1 of lines between them must be odd. If $n_1 \geqslant 3$ or for any $k \geqslant 2$, $n_k \geqslant 4$, G_2 is overlapping by Remark 2.20(v). So $n_1 = 1$ and $n_k = 2$ for all $k \geqslant 2$, and G takes the form shown in Fig. 11c. Thus for all $k \in \{1,..., r\}$, C_k is a two-legged nonoverlapping graph with at most $N-1$ vertices. By the inductive hypothesis, all the C_k are GST graphs or strings of BST graphs, so G is a string of GST graphs as well. ∎

We now turn to the four-legged case, and begin by a simple characterization of one-particle reducible four-legged graphs.

Remark 2.23. Let G be a four-legged graph and all vertices of G have an even incidence number. If G is one-particle-reducible, G is obtained from a 1PI four-legged graph G' by attaching strings of two-legged diagrams to the external legs of G', as shown in Fig. 12.

Proof. Induction on the number of vertices of G. Let G be 1P reducible and l a line such that cutting l disconnects the graph. Upon cutting l, $G-l$ falls into two connected components. Their numbers of external legs must add up o six. Since, by assumption, any subgraph of G must have an even number of external legs, one of them must be four-legged and the other one two-legged. Apply the inductive hypothesis to the four-legged subgraph, then the statement follows for G itself. ∎

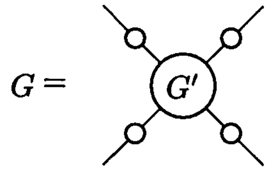

Fig. 12. General form of a one-particle reducible four-legged graph.

Definition 2.24. (i) A GSF graph is a graph G with four external legs, all joining to a single vertex v_1 of G, such that upon deletion of two of the external legs, G becomes a GST graph.

(ii) A dressed bubble chain (DBC) of length $r \geqslant 0$ is a four-legged graph as follows. There are $r+1$ GSF graphs $G_1, \ldots, G_{r+1}$ such that for all $i \in \{1, \ldots, r\}$, G_i is joined to G_{i+1} by exactly two strings of GST graphs, and the external legs of G_1 and G_{r+1} are connected to the external legs of G by strings of GST graphs (which may consist of only a single line).

Remark 2.25. If an external vertex v of a nonoverlapping four-legged diagram has at least two external legs, joining them to form a self-contraction gives a nonoverlapping two-legged diagram which must be a GST string. This is used in the proof of the following lemma. An example for a GSF graph is shown in Fig. 13a. The thick lines in this figure stand for strings of GST diagrams. An example of a DBC with $r=2$ is given in Fig. 13b, again denoting strings of GST diagrams by thick lines and denoting GSF graphs by four-legged vertices with a box. An example with $r=1$ where all vertices and lines are drawn is shown in Fig. 13c. A DBC of length $r=0$ is a GSF with strings of GST diagrams attached to the external vertex of the GSF.

Lemma 2.26. Let G be a connected graph whose vertices all have an even incidence number, and with number of external lines $E(G)=4$. If G is nonoverlapping, then G is a DBC. More precisely, let $V_E \in \{1, 2, 3, 4\}$ be the number of external vertices of G (a vertex v is called external if an

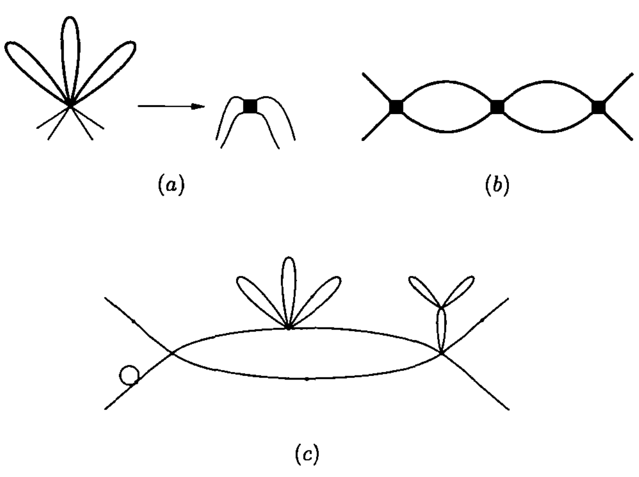

Fig. 13. Examples of GSF and DBC diagrams.

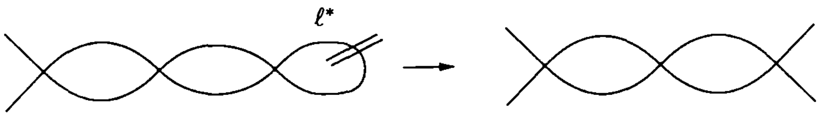

Fig. 14. Cutting a line of a GST diagram can produce a DBC.

external leg of G joins to v). If G is 1PI and nonoverlapping, then $V_E \leqslant 2$ with G a GSF for $V_E = 1$ and a DBC of length $r \geqslant 1$ for $V_E = 2$.

Proof. For $V_E \leqslant 3$, one of the external vertices, v_1, must have at least $E_1 \geqslant 2$ external lines going in. Two of the external lines of v_1 can be joined to a self-contraction l^*. By Remark 2.20(iv), the resulting two-legged graph G^* is still nonoverlapping, so by Lemma 2.22, it is a string of GST graphs. Cutting l^*, we see that G itself is a DBC. This is proven by the same induction process as is used to define GST. See Figure 14 for an example of how a DBC is generated when l^* is cut. If $E_1 \geqslant 3$, G is a DBC of length $r = 0$. If $V_E = 3$, the two-legged graph G^* constructed from G has two external vertices. Since it is nonoverlapping, it must be 1P reducible by Lemma 2.22 and Definition 2.21(iii), so G is also 1P reducible. Thus $V_E = 3$ is impossible if G is 1PI.

If $V_E = 4$, we use Remark 2.23 to decompose G into the 1PI graph G' and the strings of two-legged subdiagrams attached to G'. By Remark 2.20(i), G' must also be nonoverlapping and the strings must consist of GST diagrams. If $V_E(G') \leqslant 2$, we know by the above that G', and hence G, is a DBC. Now, $V_E(G') = 3$ is impossible, since G' is 1PI. Thus, to complete the proof, we only have to show that $V_E(G') = 4$ is impossible as well for a four-legged nonoverlapping 1PI graph G'. So assume that $V_E = 4$, let v be an external vertex of G', and let $S = G' - v$. Since $V_E = 4$, v binds only one external leg of G', so v connects to S by an odd number of lines. Let $C_1, \ldots, C_p$ be the connected components of S. One of them must connect to v by an odd number n^* of lines. But if $n^* = 1$, G' is reducible, contrary to our assumption, and if $n^* \geqslant 3$, G' is overlapping by Remark 2.20(ii), (v), again a contradiction.

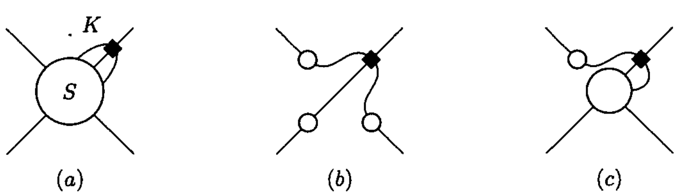

Fig. 15. The case $V_E(G') = 4$.

The alternatives are sketched in Fig. 15. The black box K consists of v, together with all connected components of S that do not contain an external vertex. In the figure, K is drawn four-legged; in general, it may have a larger incidence number. Figure 15a is the case $n^* = 3$. The two loops joining K and S overlap. Figures 15b and 15c are cases where $n^* = 1$. The figure can be disconnected by cutting a line leaving K. ∎

2.5. Decomposition of the Tree of a Labeled Graph

We now consider labeled graphs and show how to decompose the associated tree into subtrees corresponding to overlapping and nonoverlapping graphs. It was mentioned in the motivation of the classification of graphs into overlapping and nonoverlapping ones that the bound for the value of overlapping graphs contains as a factor the function I_2 defined in (1.34). As discussed in Section 1, this factor arises because the propagators of scale $j < 0$ are supported in a shell of thickness M^j near the Fermi surface, and the intersection of such a shell with its translate by some momentum $\mathbf{p}$ is transversal for all $\mathbf{p}$ outside a set whose volume shrinks with the thickness of the shell. Therefore, the arguments η_k in I_2 will be M^{j_k}, where j_k are scales of the lines involved. The volume improvement factor might arise only at a relatively high scale, and to exploit it as much as possible, it is therefore very important in our analysis to keep track of the scale at which this volume improvement factor arises. We do this by decomposing the tree of the labeled graph G^J into maximal subtrees corresponding to nonoverlapping subgraphs.

We start with an example to illustrate the idea behind the procedure. In Fig. 16, a graph with scale assignments $0 > h' > h > j$ is shown from top to bottom on decreasing scales. The interaction lines appear only on scale zero. On the lowest scale j (root scale), the graph is nonoverlapping, since all lines of higher scale are collapsed into effective vertices. On scale $h > j$, the graph is overlapping, and the volume improvement factor arises at scale h in this example. In general, the strategy will be to go from lower to higher scales (from bottom to top in Fig. 16), resolving (i.e., expanding) the effective vertices until either scale zero or a scale on which the graph overlaps is reached. With a properly chosen spanning tree for the graph, the volume gain is then extracted.

Definition 2.27. (i) Let G^J be a labeled graph with tree t. For a (connected) subtree t' of t (we shall denote this as $t' \subset t$), rooted at a fork $\phi_{t'}$, define the projected graph $\tilde{G}(t')$ as a quotient graph of $G^J_{\phi_{t'}}$ as follows. If $f'' \notin t'$ is a fork directly above t', i.e., there is a fork $f' \in t'$ such that $\pi(f'') = f'$, replace $G^J_{f''}$ by a vertex with the same external legs as $G^J_{f''}$, and with vertex function $Val(G^J_{f''})$. The lines in subgraphs G^J_f with $f \in t'$ join

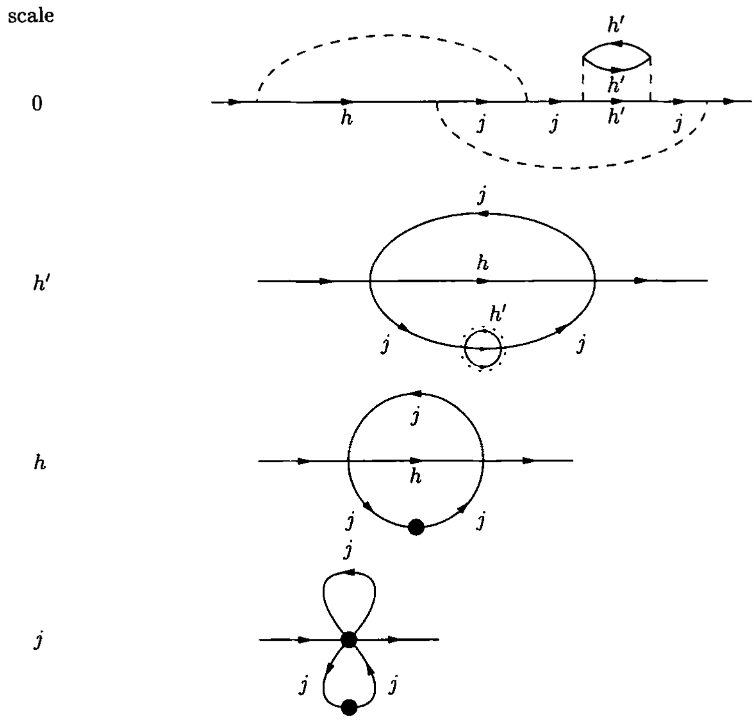

Fig. 16. A labeled graph at several scales.

these vertices to form the graph $\tilde{G}(t')$ (leaves of t that are also leaves of t' remain the same vertices they were before).

(ii) For a subset A of the set of forks and leaves of t, define

$$\sigma(A) = \{f' \in t \backslash A: f' \text{ fork}, \exists f \in A: \pi(f') = f\}$$

$$\mathscr{A}(A) = \{f' \in t \backslash A: f' \text{ fork}, \exists f \in A: f' \geqslant f\}$$

Thus $\mathscr{A}(A)$ is the set of all forks of $t \backslash A$ that are above A and $\sigma(A)$ is the set of all forks of $t \backslash A$ that are immediately above A.

(iii) For $f \in t$, denote by t_f the subtree of t rooted at f that contains all forks and leaves in $\mathscr{A}(\{f\})$.

Remark 2.28. (i) $\tilde{G}(t')$ is the graph where all subdiagrams belonging to forks above t' are collapsed to effective vertices, and where all subdiagrams belonging to forks in t' remain subdiagrams.

(ii) $\tilde{G}(t')$ is connected.

(iii) The mapping $G^J_{\phi_{t'}} \to \tilde{G}(t')$ also acts naturally on sets ϑ of lines of $G^J_{\phi_{t'}}$. Those lines in ϑ corresponding to t'-forks are left unchanged. All others are absent in the projection. The projections of paths L, etc., will be denoted as $\tilde{L}$, or $\tilde{L}(t')$ if necessary. Note that the projection of a path need not be a path in the sense of Definition 2.17 because it may visit a vertex more than once and hence fail to be injective.

(iv) t' may be trivial, that is, consist only of its root fork; then we write $t' = \phi_{t'}$ and $\tilde{G}(t') = \tilde{G}(\phi)$.

To do the tree decomposition, we need some more facts about nonoverlapping graphs, which we state in the following lemma. If G is a graph and H a connected subgraph with $2m$ external legs, we denote by G/H the quotient graph obtained by replacing H by a vertex with incidence number $2m$. In our convention, external legs of a connected graph are not counted as lines of the graph, and the statement that two subgraphs A and B of a given graph are disjoint means that they share no vertex (so an external vertex of A may be connected to an external vertex of B by a line which belongs neither to A nor to B).

Lemma 2.29. Let G be a connected graph.

(i) Let G have only vertices with an even incidence number, let T be a connected two-legged subgraph of G, and assume that G/T is nonoverlapping. Then

$$G \text{ is overlapping} \Leftrightarrow T \text{ is overlapping}$$

(ii) Let G_1 and G_2 be disjoint connected subgraphs, and assume that $\tilde{G}_1 = G/G_1$ and $\tilde{G}_2 = G/G_2$ are nonoverlapping. Then G is nonoverlapping.

Proof. (i) "$\Leftarrow$" is obvious by Remark 2.20(iii). "$\Rightarrow$": There are two independent overlapping loops K and L in G. Since G/T is nonoverlapping, their traces must differ in T. If T were nonoverlapping, T would be a string of GST, and by the structure of GST graphs and the condition that any path may visit a given vertex at most once, both K and L would have to step over the same lines in T. So then $\theta(K) = \theta(L)$, which is a contradiction.

(ii) Assume G to be overlapping. Then there are independent loops K and L such that the set of lines which are part of both loops is not empty. Thus there exist "splitting points," which are vertices as follows: v is a splitting point if v is endpoint of a line l_0 that is part of both K and L, and of lines k and l such that k is a line of $\theta(K)$ but not of $\theta(L)$, and l is a line of $\theta(L)$, but not of $\theta(K)$. In other words, a splitting point is a vertex at which the two paths deviate after going over the same line(s). Let v be such a splitting point, and l_0, l, and k be as defined above. Also, denote the second endpoint of l_0 by w.

If $v \in G_1$, we will construct loops K_2 and L_2 in $\tilde{G}_2$ as follows. First we reparametrize K and L (using the shifts and inversions described in Remark 2.18) so that they start at w and the first line is l_0, and the second is k for K and l for L, etc. Since $v \in G_1$, and since G_1 and G_2 are disjoint, none of l_0, l, and k can be in G_2, so l_0, l, and k are all in $\tilde{G}_2$.

If w is in G_2, we take K_2 to be the restriction of K up to the first point when a vertex of G_2 is hit by K; this is a loop in $\tilde{G}_2$. L_2 is defined similarly. By construction, $\theta(K_2)$ contains k, but not l, and $\theta(L_2)$ contains l, but not k, so these loops are independent, and both contain l_0. So $\tilde{G}_2$ is overlapping, which is a contradiction.

If w is not in G_2, we take K_2 to be identical to K up to the first point where K hits G_2; then we continue it to be K from the last time K visits a vertex of G_2 (if K does not visit G_2, $K_2 = K$). L_2 is defined similarly. Again, these loops are independent, and overlap at l_0, which is a contradiction.

If $v \notin G_1$, we construct loops K_1 and L_1 in $\tilde{G}_1$ starting again at w, and going over l_0 and k or l, this time taking out the parts between first and last visits of G_1, to avoid multiple visits at the vertex of $\tilde{G}_1$ that replaces G_1. Since v is not in G_1, the lines l_0, l, and k are all in $\tilde{G}_1$, so K_1 and L_1 are again independent overlapping loops in $\tilde{G}_1$. This contradicts the assumption that $\tilde{G}_1$ is nonoverlapping. ∎

Remark 2.30. If the vertices are allowed to have odd incidence numbers, (i) does not hold, as can be seen from the graph in Fig. 17 (the subgraph H is the part of G inside the dashed circle).

Lemma 2.31. Let G^J be a labeled graph, t its tree, and $f \in t$ a fork.

(i) Let $f_1, \ldots, f_n \in t$ be forks or leaves such that $\pi(f_i) = f$ $\forall i$, and assume that for all $i \in \{1, \ldots, n\}$, $\tilde{G}(^{f_i}_{f})$ is nonoverlapping. Then

$$\tilde{G}\begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ & & f & \end{pmatrix}$$

is nonoverlapping as well.

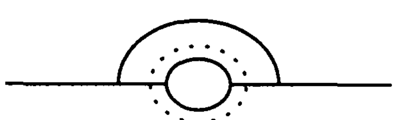

Fig. 17. A graph with vertices with odd incidence number.

(ii) Let $\tilde{G}(f)$ be nonoverlapping. Then there is a unique maximal tree $\tau_f \subset t$, rooted at f, such that $\tilde{G}(\tau_f)$ is nonoverlapping, i.e., if τ is such that $\tilde{G}(\tau)$ is nonoverlapping and such that $\phi_\tau = f$, then $\tau \subset \tau_f$.

(iii) There is

$$\mathcal{N} \subset \{\phi\} \cup \{f\colon E(G_f^J) = 2,\ \tilde{G}(f) \text{ 1PI and nonoverlapping}\} \tag{2.77}$$

such that (a) $\phi \in \mathcal{N}$ if and only if $\tilde{G}(\phi)$ is nonoverlapping, (b) if $f, f' \in \mathcal{N}$ with $f \neq f'$, then τ_f and $\tau_{f'}$ are disjoint, and (c) if $f \notin \mathcal{N}$ but $E(G_f^J) = 2$ and $\tilde{G}(f)$ is 1PI and nonoverlapping, then there is an $f' \in \mathcal{N}$ with $\tau_f \subset \tau_{f'}$.

Here τ_f, $\tau_{f'}$ are the maximal trees associated to f, f' in part (ii).

Proof. (i) follows as in Lemma 2.29(ii).

(ii) Let $S = \{t' \subset t_f\colon \tilde{G}(t') \text{ nonoverlapping}\}$. Since $\tilde{G}(f)$ is nonoverlapping, $S \neq \varnothing$. Build up the tree τ_f recursively as follows: for all forks or leaves $f_1,\ldots, f_n$ with $f = \pi(f_i)$, add $\genfrac{}{}{0pt}{}{f_i}{f}$ to τ_f if $\tilde{G}(\genfrac{}{}{0pt}{}{f_i}{f})$ is nonoverlapping [note that if σ is a leaf, then $\tilde{G}(\genfrac{}{}{0pt}{}{\sigma}{f})$ is always nonoverlapping if $\tilde{G}(f)$ is nonoverlapping, since $\tilde{G}(\genfrac{}{}{0pt}{}{\sigma}{f}) = \tilde{G}(f)$]. The resulting tree

$$\tilde{G}\left(\begin{matrix} f_{i_1} & f_{i_2} & \cdots & f_{i_k} \\ & f & & \end{matrix}\right)$$

is then nonoverlapping by (i). If for all forks $f_1,\ldots, f_n$, $\tilde{G}(\genfrac{}{}{0pt}{}{f_i}{f})$ is overlapping, then

$$\tau_f = \begin{matrix} v_1 \cdots v_b \\ f \end{matrix}$$

where $v_1,\ldots, v_b$ are the leaves with $\pi(v_i) = f$, or $\tau_f = f$ if $b = 0$, and the process stops. Otherwise, repeat the procedure for every $f' \in \{f_{i_1},\ldots, f_{i_k}\}$ that is a fork, add branches $\genfrac{}{}{0pt}{}{f''}{f_{i_l}}$ if the corresponding graph is nonoverlapping, add all branches to leaves, and stop if there are no forks with nonoverlapping graphs $\tilde{G}(\genfrac{}{}{0pt}{}{f''}{f_{i_l}})$. Repeating this, the process ends after a finite number of steps. It is obvious by construction that the tree so obtained is maximal in S and therefore unique.

(iii) Put ϕ into $\mathcal{N}$ if $\tilde{G}(\phi)$ is nonoverlapping, and in that case construct τ_ϕ using (ii). Let $M_2(G^J) = \{f \in t\colon f > \phi,\ \tilde{G}(f) \text{ is nonoverlapping, 1PI and two-legged}\}$. We construct $\mathcal{N}$ by induction on the number of forks N of $M_2(G^J)$. If $N = 0$, i.e., $M_2(G^J) = \varnothing$, then $\mathcal{N} = \varnothing$ or $\mathcal{N} = \{\phi\}$, depending on whether $\tilde{G}(\phi)$ is overlapping or not. Let $N \geqslant 1$ and assume that the family has been constructed for all $N' \leqslant N - 1$. Let $\{f_1,\ldots, f_n\}$ be the set of all minimal forks of $M_2(G^J)$ (in the partial ordering of t). For all k, construct τ_{f_k} by (ii). Because of the tree structure, we can consider each k separately. Let $g = f_k$ for some k. Now, $\mathscr{A}(\tau_g)$ is a disjoint union of trees rooted at forks $f \in \sigma(\tau_g)$ [or $\mathscr{A}(\tau_g) = \varnothing$, in which case we are done with g]. Each of these trees has $N' \leqslant N-1$ forks in M_2, so the inductive hypothesis applies. Add to $\mathscr{N}$ the forks that have been selected by the inductive hypothesis from each of the trees. This really gives a family of disjoint trees in the sense that no element of $\mathscr{N}$ is directly above a fork in a tree τ_f of $\mathscr{N}$, as is implied by the remark following this lemma.

Suppose now that $f \in t \backslash \mathscr{N}$, but $E(G_f^J) = 2$ and $\tilde{G}(f)$ is 1PI and nonoverlapping. By the construction of $\mathscr{N}$ the set $\{f'' \in \mathscr{N}: f'' < f\}$ is nonempty. Let f' be the maximal element of this set. Also by construction $f \in \tau_{f'}$. To complete the proof it suffices to show that if $\tau_f \not\subset \tau_{f'}$, then the tree $\tau_{f'}$ is not maximal in the sense of (ii). To see this, first observe that $\tilde{G}((\tau_{f'})_f)$ is nonoverlapping by Remark 2.20(i). So the maximality of τ_f implies $(\tau_{f'})_f \subset \tau_f$, which in turn implies $\tilde{G}(\tau_f \cup \tau_{f'})$ is obtained by replacing the two-legged nonoverlapping subgraph $\tilde{G}((\tau_{f'})_f)$ of $\tilde{G}(\tau_{f'})$ by the two-legged nonoverlapping graph $\tilde{G}(\tau_f)$. Remark 2.20(ii) and the following remark ensure that $\tilde{G}(\tau_f \cup \tau_{f'})$ is nonoverlapping. So the assumption that $\tau_{f'} \neq \tau_f \cup \tau_{f'}$ contradicts the maximality of $\tau_{f'}$. Thus $\tau_f \subset \tau_{f'}$. ∎

Remark 2.32. Let G^J be a labeled graph of our model, $t(G^J)$ its tree, and let τ be a subtree of t such that $\tilde{G}(\tau)$ is nonoverlapping. Let f be a fork directly above τ, i.e., $f \in \sigma(\tau)$, such that $E(G_f) = 2$. Then

$$\tilde{G}\begin{pmatrix} f \\ | \\ \tau \end{pmatrix} \text{ overlapping} \Leftrightarrow \tilde{G}(f) \text{ overlapping} \tag{2.78}$$

Proof. Apply Lemma 2.29(i). ∎

Remark 2.33. Note that Remark 2.32 holds only for two-legged subdiagrams. For example, the graph at scale j in Fig. 16 is not overlapping, while that at scale h is overlapping. Nonetheless, to go from the graph at scale j to that at scale h, one replaces the six-legged vertex by the tree diagram shown in Fig. 18, which has no loops and hence is not

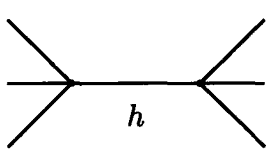

Fig. 18. The six-legged scale h subgraph of Fig. 16.

overlapping. So, it is possible to replace one vertex in a nonoverlapping graph by a nonoverlapping subgraph and produce an overlapping graph. This plays a role for estimates of derivatives.

2.6. Improved Power Counting

We now extract the volume improvement factor in the value of any graph that overlaps at some scale, and use it to show an improved power counting bound that holds for every such graph. We also give a natural routing prescription for the external momentum suited to bounding derivatives. In this section, let $G \in Gr(n, m; m_1,..., m_n)$, $J: l \mapsto j_l$ be a labeling of G and $Val(G^J)(C, \mathscr{U}_{v_1},..., \mathscr{U}_{v_n})$ be given by (2.54). A loop basis for the graph is a basis for $H^1(G, \mathbb{Z})$. Since for an overlapping graph, the two overlapping loops (in the sense of Definition 2.17) define linearly independent cycles, we may use both of them as basis elements. Recall that there is a natural basis for $H^1(G, \mathbb{Z})$ associated to any spanning tree T for G. It contains one loop for each element of $L(G)\backslash L(T)$. The loop associated to $l \in L(G)\backslash L(T)$ consists of l and the path in T joining the ends of l. Also recall that T is consistent with J if $T \cap \tilde{G}(t_f)$ is a spanning tree for $\tilde{G}(t_f)$ for all forks $f \in t_\phi$.

By definition, a graph G is overlapping if there exist two independent loops in G which share a line. *A priori*, the specific loops determined by a spanning tree for G are not required to overlap. But of course they do. This is proven in the following lemma.

Lemma 2.34. (i) If G has a spanning tree T without any associated overlapping loops, then no spanning tree of G has any associated overlapping loops.

(ii) If G is overlapping and T is an arbitrary spanning tree of G, then there are two overlapping loops associated to lines l_1 and $l_2 \in L(G)\backslash L(T)$.

Proof. (i) Let T be a spanning tree for G such that all of the loops $L_1,..., L_n$ associated to T do not overlap each other. Define $T_{NL} = G\backslash \bigcup_{i=1}^n L_i$. Then (i) is a consequence of (a) if $l \in T_{NL}$, then $G - l$ is not connected, and (b) for each $1 \leqslant i \leqslant n$, every spanning tree for G must contain all of L_i save exactly one line.

Now, (a) and (b) imply that any spanning tree for G must consist of G minus exactly one line from each of $L_1,..., L_n$.

Proof of (a). First note that $T_{NL} \subset T$ because, by definition, every line of G that is not in T generates one of $L_1,..., L_n$. If $G - l$ is connected, there is a path in $G - l$ that joins the two vertices at the ends of l. There is always such a path that is also contained in T, because T contains all of

G save one line from each of $L_1,\dots, L_n$. If the path uses the missing line from L_i, we can always replace the missing line by the rest of L_i. Hence l union the path is a loop in T, which is impossible.

Proof of (b). Delete two lines l_1, l_2 from $\theta(L_i)$ [$\theta(L)$ is the subgraph corresponding to L; see Definition 2.17(iii)]. Then $\theta(L_i) - l_1 - l_2$ consists of two connected pieces A_1, A_2. In the event that l_1 and l_2 are nearest neighbors on L_i, A_1 and/or A_2 is a trivial graph consisting of a single vertex. Suppose that there is a path P in $G - l_1 - l_2$ connecting a vertex v_1 of A_1 to a vertex v_2 of A_2. We can assume without loss of generality that this path contains no lines of L_1. As in the proof of (a), we can also arrange for the path to be contained in T. One of l_1 and l_2 must be in T, so we can construct a loop in T using P and part of L_1. This is impossible, so P cannot exist. So no spanning tree for G can be contained in $G - l_1 - l_2$.

(ii) It suffices to construct one spanning tree T for G that has two overlapping loops associated, because by (i), any other spanning tree for G will then have the same property. Let L_1 and L_2 be independent overlapping loops in G. Let l_1 be a line in L_1 that is not in L_2. The line l_1 exists because if $\theta(L_1) \subset \theta(L_2)$, then either $\theta(L_1) = \theta(L_2)$ or L_2 is self-intersecting. Put $\theta(L_1) - l_1$ in T. Note that, regardless of how we complete T, the loop associated to l_1 will always be L_1. Let l_2 be a line that is in L_2, but not in L_1. Denote by v_1 and v_2 the vertices at the ends of l_2. Add to T the unique connected subgraph of $\theta(L_2)$ that does not contain l_2, that has v_1 as one terminating vertex, that has a vertex w_1 of $\theta(L_1)$ as its other terminating vertex, and that contains only one vertex of $\theta(L_1)$. Similarly, add to T the unique connected subgraph of $\theta(L_2)$ that does not contain l_2, that has v_2 as one terminating vertex, that has a vertex w_2 of $\theta(L_1)$ as its other terminating vertex, and that contains only one vertex of $\theta(L_1)$. Note that the two pieces of L_2 that have just been added to T contain no lines of L_1 and that $w_1 \neq w_2$, because L_2 must overlap L_1 and cannot be self-intersecting. Regardless of how we complete T, the loop M_2 associated to l_2 will contain l_2, continue along L_2 from v_1 to w_1, continue along L_1 from w_1 to w_2, and finally continue along L_2 from w_2 to v_2. Thus the loops associated to l_1 and l_2 overlap. Complete T any way you like. ∎

Lemma 2.35. Let G be overlapping. Let J be an assignment of scales to G.

(i) Let τ_ϕ be the maximal subtree of $t(G^J)$ rooted at ϕ for which $\tilde{G}(\tau_\phi)$ is nonoverlapping. Let $j^* = \min\{j_f\colon f \in \sigma(\tau_\phi)\}$. Then for any tree T consistent with J there is a line $l^* \in T$ with $j_{l^*} \leqslant j^*$ which is contained in two independent loops associated to lines l_1 and $l_2 \in L(G)\backslash L(T)$. In the case that $\tilde{G}(\phi)$ is overlapping, $j^* = j_\phi$. In the assignment of momenta to lines of G given by T,

$$p_{l^*} = \pm p_{l_1} \pm p_{l_2} + Q \tag{2.79}$$

where Q is a linear combination of loop and external momenta independent of p_{l_1} and p_{l_2}.

(ii) Assume that the propagators assigned to the lines of G satisfy

$$|C_{j_l}(p_0, e(\mathbf{p}))| \leqslant z_l M^{-j_l+2} \, 1(|ip_0 - e(\mathbf{p})| \in [M^{j_l-2}, M^{j_l}]) \tag{2.80}$$

with factors $z_l > 0$. Let K_0 be as in Lemma 2.3(ii), ε be as in Proposition 1.1, A be as in Lemma 2.3(ii), and let

$$K_1 = C_{\text{vol}} \frac{u_0^2}{A^2} \tag{2.81}$$

Then

$$|Val(G^J)|_0 \leqslant K_1 \prod_{l \in L(G)} (4K_0 z_l) \prod_{v \in V(G)} |\mathcal{U}_v|_0 \, M^{\varepsilon j_{l^*}} M^{D_\phi j_\phi} \times \prod_{f > \phi} M^{D_f(j_f - j_{\pi(f)})} \tag{2.82}$$

Proof. (i) Let T be any tree consistent with J. For example, T may be built by first building spanning trees for the topmost forks of $t(G^J)$, then extending these to spanning trees for the next level of forks of $t(G^J)$, and so on. Let b obey $j_b = \min\{j_f \colon f \in \sigma(\tau_\phi)\}$. Then $\tilde{T}(\tau_\phi \cup \{b\}) \equiv T \cap \tilde{G}(\tau_\phi \cup \{b\})$ is a spanning tree for $\tilde{G}(\tau_\phi \cup \{b\})$, because if you collapse a *connected* subgraph of a tree, you get another tree. By the maximality of τ_ϕ, $\tilde{G}(\tau_\phi \cup \{b\})$ is overlapping. By Lemma 2.34(ii), any spanning tree for any overlapping graph has associated at least two overlapping loops. So there is an $l^* \in \tilde{G}(\tau_\phi \cup \{b\})$ that is in two independent loops $\tilde{L}_1$ and $\tilde{L}_2$ associated to $\tilde{T}(\tau_\phi \cup \{b\})$. These loops expand to two independent loops L_1 and L_2 associated to T, both of which contain l^*. Then (2.79) follows for l^* because any line that is part of the loop L_i has the momentum p_{l_i} flowing through it, i.e., the linear combination of momenta making up p_{l^*} contains summands $\pm p_{l_1}$ and $\pm p_{l_2}$, the $\pm$ depending on the relative orientation of the lines.

(ii) After fixing of the momenta on the lines of T, the expression (2.54) for $Val(G)$ becomes

$$Val(G^J)(\eta_1,..., \eta_{2m-1}, \beta_{2m})$$
$$= \sum_{\text{spins }\alpha} \int \prod_{l\in L(G)\backslash L(T)} d^{d+1}p_l \prod_{l\in L(G)} (C_{j_l}(p_l))_{\alpha_l\alpha_l'}$$
$$\times \prod_{i=1}^{n} (\mathscr{U}_{v_i}(p_1^{(i)},..., p_{2m_i-1}^{(i)}))_{\alpha_1^{(i)},..., \alpha_{2m_i}^{(i)}} \tag{2.83}$$

where $\eta_k = (q_k, \beta_k)$ and $2m_i = E_{v_i}$ and the momenta on lines and in the vertex functions $\mathscr{U}_v$ match up according to the fixing of the momenta described in (i), and for each $t \in L(T)$, p_t is a linear combination of the loop momenta $(p_l)_{l\in L(G)\backslash L(T)}$ and the external momenta $q_1,..., q_{2m-1}$.

We bound the spin sum at both ends of every line $l \in L(G)$ by a factor 2 times the maximum over spins and take the sup norm of all $\mathscr{U}_{v_i}$, to get

$$|Val(G)(C, \mathscr{U}_{v_1},..., \mathscr{U}_{v_n})|_0 \leqslant 4^{|L(G)|} X \prod_{k=1}^{n} |\mathscr{U}_{v_k}|_0 \tag{2.84}$$

where

$$X = \sup_{q_1,..., q_{2m-1}} \int \prod_{l\in L(G)\backslash L(T)} d^{d+1}p_l \prod_{l\in L(G)} |C_{j_l}((p_l)_0, e(\mathbf{p}_l))| \tag{2.85}$$

By hypothesis, upon integration over the $(p_l)_0$,

$$X \leqslant Y\Big(\prod_{l\in L(G)} z_l M^{-j_l+2}\Big) \prod_{l\in L(G)\backslash L(T)} (2M^{j_l}) \tag{2.86}$$

where

$$Y = \sup_{q_1,..., q_{2m-1}} \int \prod_{l\in L(G)\backslash L(T)} (d^d\mathbf{p}_l\, 1(|e(\mathbf{p}_l)| \leqslant M^{j_l}))$$
$$\times \prod_{l\in T} 1(|e(\mathbf{p}_l)| \leqslant M^{j_l}) \tag{2.87}$$

Ordinary power counting, we would obtain Lemma 2.4(i) by omitting the last product over $l \in T$. Improved power counting is obtained by keeping only one factor, that from $l = l^* \in T$, of this product, to use the volume improvement estimate, Proposition 1.1. Applying (i), integrating over the loop momenta p_{l_1} and p_{l_2} first, and recalling (1.34), we obtain

$$Y \leqslant I_2(M^{j_{l_1}}, M^{j_{l_2}}, M^{j_{l^*}}) \int \prod_{\substack{l\in L(G)\backslash L(T)\\ l\notin\{l_1, l_2\}}} d^d\mathbf{p}_l\, 1(|e(\mathbf{p}_l)| \leqslant M^{j_l}) \tag{2.88}$$

By Proposition 1.1 and Lemma 2.3(ii),

$$Y \leqslant C_{\mathrm{vol}} M^{\varepsilon j_{l^*}} M^{j_{l_1}} M^{j_{l_2}} \prod_{\substack{l \in L(G)\backslash L(T) \\ l \notin \{l_1, l_2\}}} \frac{A}{u_0} M^{j_l}$$

$$\leqslant K_1 M^{\varepsilon j_{l^*}} \prod_{l \in L(G)\backslash L(T)} \frac{A}{u_0} M^{j_l} \tag{2.89}$$

We insert this bound for Y, use $|L(G)\backslash L(T)| \leqslant |L(G)|$, and $A/u_0 \geqslant 1$, and reorder the product over scales by the usual telescope formula

$$j_l = j_\phi + \sum_{\substack{f > \phi \\ l \in G_f}} (j_f - j_{\pi(f)}) \quad \blacksquare$$

Remark 2.36. Apart from a constant, the improved power counting bound is the ordinary power counting bound times an improvement factor $M^{j^* \varepsilon}$, where j^* is the scale at which the graph overlaps. By Lemma 2.3(i), the propagators C_j given by (2.17) satisfy the hypothesis of (ii) with $z_l = 1$. Derivatives with respect to p or e satisfy a bound with $z_l = \mathrm{const} \cdot M^{-j_l}$ by Lemma 2.3(iii).

We now want to prove that for any labeled graph G^J with (scale) tree $t(G^J)$, there is a spanning tree such that the external momentum does not enter any of the lines in $\tilde{G}(\tau_f)$, for all $f \in \mathcal{N}$. We first explain why there is anything to prove. Any two-legged 1PI nonoverlapping graph has only one external vertex v_1. The external momentum can trivially avoid all internal lines of such a graph. However, even if $\tilde{G}(f)$ is nonoverlapping, $G(t_f)$ may be overlapping. In fact, the image of a poorly chosen spanning tree for G under the projection onto $\tilde{G}$ may not even be a tree. Consider, for example, the graph drawn in Fig. 16. If the leftmost line carrying scale j is in the spanning tree of G at scale zero (top of the figure), what remains of T in the projection of G on scale j (bottom of the figure) is certainly not a tree graph. The way to avoid this problem is to start at the bottom, i.e., at root scale, to construct a spanning tree for $\tilde{G}(\phi)$, and then go upward on scales, constructing spanning trees for all the subgraphs that appear as effective vertices, and combine them to a spanning tree for G using the following simple fact.

Remark 2.37. Let G be a graph, G' a connected subgraph of G, and $\tilde{G}$ the quotient graph of G obtained by replacing G' by a vertex. Let T' be a spanning tree for G' and $\tilde{T}$ a spanning tree for $\tilde{G}$. Note in particular that T' necessarily consists only of *internal* lines of G'. Let $T = T' \cup \tilde{T}$. Then T is a spanning tree of G.

Lemma 2.38. Let G^J be a labeled two-legged 1PI graph and let $(\tau_f)_{f \in \mathcal{N}}$ be the family of subtrees of Lemma 2.31(iii). Then there is a spanning tree of G such that for all $f \in \mathcal{N}$, the external momentum enters in no line of $\tilde{G}(\tau_f)$, and such that for all $f \in \mathcal{N}$, there is an improvement factor $M^{\varepsilon j_f^*}$ with $j_f^* \leqslant \min\{j_b\colon b \text{ leaf of } \tau_f\}$.

Proof. We first construct a suitable spanning tree for $\tilde{G}(\tau_f)$, for any $f \in \mathcal{N}$. This is easy. On each loop of $\tilde{G}(\tau_f)$ delete a line l' of lowest scale. Since $\tilde{G}(\tau_f)$ is a GST graph, this does not disconnect it. It leaves a tree T_f, which is already the desired spanning tree for $\tilde{G}(\tau_f)$. In particular, it is consistent with τ_f. This means the following. Let f' be a fork of τ_f and $t_{f'}$ the subtree of τ_f consisting of f' and all forks of τ_f above f'. Then $T_f \cap \tilde{G}(t_{f'})$ is also a spanning tree for $\tilde{G}(t_{f'})$. To see consistency, it suffices to check that $T_f \cap \tilde{G}(t_{f'})$ connects all pairs of vertices v, v' of $\tilde{G}(t_{f'})$, because clearly $T_f \cap \tilde{G}(t_{f'})$ can contain no loops. As $\tilde{G}(t_{f'})$ is connected, it contains some path from v to v'. The only problem is that this path may use the one line l' of some loop L that is omitted from T_f. But because $l' \in \tilde{G}(t_{f'})$ and $j_{l'} \leqslant j_{l''}$ for all $l'' \in L$, we necessarily have $L \subset \tilde{G}(t_{f'})$. But then we may use $L \backslash l' \subset T_f$ instead of l' in the path.

Since the loops determined by T_f do not overlap and $\tilde{G}(\tau_f)$ is 1PI, all lines on the same loop carry precisely the loop momentum $p_{l'}$. The external momentum enters only in the vertex function of the one external vertex, but not in any internal line of $\tilde{G}(\tau_f)$. Now we combine them, going upward from the lowest forks $f \in \mathcal{N}$. Choose a leaf b of τ_f such that $j^* = j_b$ is minimal. By the maximality of τ_f, $G'_f = \tilde{G}(\,^{b}_{\tau_f})$ is overlapping. Grow a spanning tree for $\tilde{G}(b)$. Combine it with the spanning tree T_f of $\tilde{G}(\tau_f)$ by Remark 2.37 to get a spanning tree for G'_f. Lemma 2.35 applies, so there is a volume improvement factor on scale j^* or below. To get a spanning tree for G, we do the above procedure for all $\tilde{G}(\tau_f)$, $f \in \mathcal{N}$, then choose an appropriate spanning tree for the remaining subgraphs of G and put them together using Remark 2.37 to obtain a spanning tree T for G. This is possible because, by Lemma 2.31(iii) and Remark 2.32, all the G'_f's are disjoint. ∎

Remark 2.39. Note that the external momentum does enter internal lines of nonoverlapping 1PI two-legged graphs if vertices with odd incidence number are there; see, e.g., the graph of Remark 2.30.

Theorem 2.40. (Improved power counting). Let G^J be a labeled graph contributing to the sum (2.72) for $G^I_{2m,r}$, let t be the tree associated to G^J, and let ϕ be its root. Let $\mathcal{N}$ be as in Lemma 2.31(iii) and for $f \in \mathcal{N}$, let

$$j^*(f) = \min\{j_b\colon b \text{ leaf of } \tau_f\} \tag{2.90}$$

Then

$$|Val(G^J)|_0 \leqslant (4K_0)^{|L(G)|} \, (K_1 M^{\varepsilon j})^{1(\phi \notin \mathcal{N})} \prod_{f \in \mathcal{N}} (K_1 M^{\varepsilon j^*(f)})$$

$$\times M^{jD_\phi} \prod_{\substack{f \in t \\ f > \phi}} M^{D_f(j_f - j_{\pi(f)})}$$

$$\times \prod_{v \text{ two-legged}} |\theta_v|_0 \prod_{v \text{ four-legged}} |F_v|_0 \tag{2.91}$$

Proof. Choose the spanning tree of Lemma 2.38, fix the momenta, collect the improvement factors given in Lemma 2.38, going upward from root scale, and do not forget the one at root scale if the graph is overlapping on root scale, that is, $\phi \notin \mathcal{N}$. This works because the higher $\tilde{G}(\tau_f)$ appear as vertex functions in the lower ones, so that one can indeed apply Lemma 2.35 separately for all $\tilde{G}(\tau_f)$, $f \in \mathcal{N}$, and because Remark 2.32 assures that no improvement factors are counted twice. ∎

If $\tilde{G}(\phi)$ is nonoverlapping on root scale, then there is no improvement factor $M^{\varepsilon j}$. For general nonoverlapping graphs, e.g., four-legged ones, there is no further improvement without more specific assumptions on the band structure e. However, for two-legged 1PI graphs, one can use a refined bound that exploits sign cancellations to show that their root scale behavior does contain another factor of $M^{\varepsilon j}$ even if they are nonoverlapping on root scale. This bound, which we now prove, is more subtle than the previous ones and we will have to use it with care when proving the statements about the derivative with respect to e in Section 3. We first give the explicit formula for $Val(G)$ for nonoverlapping two-legged graphs.

Remark 2.41. Let G be a nonoverlapping, 1PI, two-legged graph. By Lemma 2.22, G is a GST graph. By definition, these graphs have an obvious recursive structure: let v_1 be the external vertex of G, with incidence number $2m_1$. Let $v_{i_1}, \dots, v_{i_{r_1}}$ be the vertices of G that have incidence number $\geqslant 4$ and that are on one of the self-contraction loops of v_1. By definition of a GST graph, each vertex v_{i_k} is again an external vertex of a GST (or ST) graph G_{i_k}. Choose a spanning tree for G as in Lemma 2.38. Then the value of G takes the form

$$(Val(G^J)(C, \mathcal{U}_{v_1}, \dots, \mathcal{U}_{v_n}))_{\beta\beta'}(q)$$

$$= \sum_{\alpha_1, \dots, \alpha_{2m_1-2}} \int \prod_{i=1}^{m_1-1} (d^{d+1}p_i \, (S_i(p_i))_{\alpha_{m_1-1+i}\alpha_i})$$

$$\times (\mathcal{U}_{v_1})_{\alpha_1 \cdots \alpha_{m_1-1}\beta\alpha_{m_1} \cdots \alpha_{2m_1-2}\beta'}(p_1, \dots, p_{m_1-1}, q, p_1, \dots, p_{m_1-1}) \tag{2.92}$$

where $S_i(p) \in M_2(\mathbb{C})$ are strings of subdiagrams,

$$S_i(p) = \left(\prod_{k=1}^{w_i - 1} C_{j_k}(p)\, \mathscr{P}_k T_k(p) \right) C_{j_{w_i}}(p) \tag{2.93}$$

with w_i the number of lines of the string S_i, $\mathscr{P}_k \in \{1, l, 1-l\}$, and $T_k(p)$ the kernel either of a two-legged vertex or of the (G)ST graph G_{i_k} if it is associated to one of $v_{i_1}, \ldots, v_{i_{r_1}}$. Because of (2.16), there are $j^{(i)}$ such that for all $k \in \{1, \ldots, w_i\}$, $j_k \in \{j^{(i)}, j^{(i)}+1\}$.

Lemma 2.42. Let $j < 0$, $n \geqslant 1$, $k \in \{0, \ldots, n-1\}$, $m \in \{1, \ldots, n\}$. Let $T_1 \cdots T_{n-1} \in C^2(\mathbb{R} \times \mathscr{B})$ and $g \in C(\mathbb{R} \times \mathscr{B})$. Let

$$\begin{aligned} I_j = \int_{\mathbb{R}} dp_0 \int_{\mathscr{B}} & d^d\mathbf{p}\, C_j(p_0, e(\mathbf{p}))^m\, C_{j+1}(p_0, e(\mathbf{p}))^{n-m} \\ & \times g(p_0, \mathbf{p}) \prod_{w=1}^{k} (1-l)\, T_w(p) \prod_{w=k+1}^{n-1} l T_w(p) \end{aligned} \tag{2.94}$$

Then there are constants $U_1 \leqslant U_2$, U_8 depending on M, u_0, $|u|_s$, d, and δ such that

$$|I_j| \leqslant U_1 M^{2n} M^j |g|_0 \prod_{w=1}^{k} |T_w|_1 \prod_{w=k+1}^{n-1} (|T_w|_0 M^{-j}) \tag{2.95}$$

and, if $g \in C^1(\mathbb{R} \times \mathscr{B})$,

$$\begin{aligned} |I_j| \leqslant U_2 M^{2n} M^{2j} & \prod_{w=k+1}^{n-1} (|T_w|_0 M^{-j}) \\ & \times \left(|g|_{1,j} \prod_{w=1}^{k} |T_w|_1 + |g|_{0,j} \sum_{v=1}^{k} |T_v|_2 \prod_{w \neq v} |T_w|_1 \right) \end{aligned} \tag{2.96}$$

where

$$|g|_{s,j} = \sum_{\alpha:\, |\alpha| \leqslant s} \sup\{ |\partial^\alpha g(p)| : |p_0| \leqslant M^j,\ |e(\mathbf{p})| \leqslant M^j \} \tag{2.97}$$

Proof. We change variables to (ρ, ω), as given in Lemma 2.1, and denote $\gamma(p_0, \rho, \omega) = g(p_0, \mathbf{p}(\rho, \omega))$ and $\theta_w(p_0, \rho, \omega) = T_w(p_0, \mathbf{p}(\rho, \omega))$. Then

$$\begin{aligned} \ell T_w(p) &= \theta_w(0, 0, \omega) \\ (1-\ell)\, T_w(p) &= \theta_w(p_0, \rho, \omega) - \theta_w(0, 0, \omega) \end{aligned} \tag{2.98}$$

and

$$I_j = \int_{\mathbb{R}} dp_0 \int_{\mathbb{R}} d\rho \; C_j(p_0, \rho)^m \; C_{j+1}(p_0, \rho)^{n-m}$$

$$\times \int_S d\omega \, (J\gamma)(p_0, \rho, \omega) \prod_{w=1}^{n-1} \mathscr{P}_w \theta_w(p_0, \rho, \omega) \tag{2.99}$$

Here $J(p_0, \rho, \omega) = J(\rho, \omega)$ is the Jacobian of the change of variables; see Lemma 2.1. In polar coordinates (r, φ) such that $\rho = r \sin \varphi$ and $p_0 = r \cos \varphi$, $dp_0 \, d\rho = r \, dr \, d\varphi$,

$$C_j(p_0, \rho) = \frac{f(M^{-2j} r^2)}{ire^{i\varphi}} \tag{2.100}$$

Since $m \geqslant 1$,

$$f(M^{-2j}r^2)^m \; f(M^{-2j-2}r^2)^{n-m} \leqslant f(M^{-2j}r^2) \leqslant 1(r \in [M^{j-2}, M^j]) \tag{2.101}$$

Noting that ℓT_w is independent of φ and r by (2.98), and writing the difference

$$\theta_w(r \cos \varphi, r \sin \varphi, \omega) - \theta_w(0, 0, \omega) = r\Delta_w(r, \varphi, \omega) \tag{2.102}$$

with

$$\Delta_w(r, \varphi, \omega) = \int_0^1 dt \, (\cos \varphi \, \partial_0 + \sin \varphi \, \partial_1) \, \theta_w(tr \cos \varphi, tr \sin \varphi, \omega) \tag{2.103}$$

we obtain

$$|I_j| \leqslant H_j M^{2n+1} M^j \int_S d\omega \prod_{w=k+1}^{n-1} (|\theta_w|_0 \, M^{-j}) \tag{2.104}$$

where

$$H_j = \sup_{\omega \in S} \; \sup_{r \in [0, M^j]} \left| \int_0^{2\pi} d\varphi \, e^{-in\varphi} \phi(r, \varphi, \omega) \right| \tag{2.105}$$

with

$$\phi(r, \varphi, \omega) = (J\gamma)(r \cos \varphi, r \sin \varphi, \omega) \prod_{w=1}^{k} \Delta_w(r, \varphi, \omega) \tag{2.106}$$

Bounding the φ-integral by $2\pi \, |\phi|_0$ results in the ordinary power counting bound (2.95).

But we can do better than that by being more careful about the integral over φ. After a Taylor expansion of ϕ around $r=0$,

$$\phi(r, \varphi, \omega) = \phi(0, \varphi, \omega) + r \int_0^1 dt \frac{\partial \phi}{\partial r}(tr, \varphi, \omega) \tag{2.107}$$

H_j splits into two terms. The first term contains

$$\phi(0, \varphi, \omega) = J(0, \omega)\, \gamma(0, 0, \omega) \prod_{w=1}^{k} (\cos \varphi\, \partial_0 \theta_w(0, 0, \omega) + \sin \varphi\, \partial_1 \theta_w(0, 0, \omega)) \tag{2.108}$$

which is a polynomial of degree $k \leqslant n-1$ in $e^{i\varphi}$ and $e^{-i\varphi}$, so

$$\int_0^{2\pi} d\varphi\, e^{-in\varphi} \phi(0, \varphi, \omega) = 0 \tag{2.109}$$

In the second term, we bound

$$\left| \frac{\partial \phi}{\partial r}(tr, \varphi, \omega) \right| \leqslant |J\gamma|_0 \sum_{v=1}^{k} |T_v|_2 \left(\prod_{w \neq v} |T_w|_1 \right) + |J\gamma|_1 \prod_{w=1}^{k} |T_w|_1 \tag{2.110}$$

The factor r from the Taylor expansion gives the additional M^j. Collecting the constants $|J|_1$ and others coming from the relation between γ and g, we obtain the lemma with constants U_s that depend on δ_0, $|u|_s$, u_0, and d. ∎

Remark 2.43. In the application to the value of a nonoverlapping graph, the g in Lemma 2.42 will be the vertex function $\mathscr{U}_{v_1}$, which may depend on other momenta. Replacing $|g|_0$ and $|g|_1$ by the corresponding norms of the restriction of $g(p)$ to p obeying $|p_0|$, $|e(\mathbf{p})| \leqslant M^{j+1}$ retains information about the support of the propagator C_j. As the example of Remark 2.33 shows, this information is necessary for volume improvement. In fact, Lemma 2.35 applies to that expression, in which the string S_1 is replaced by a propagator that satisfies the hypothesis of Lemma 2.35(ii) with $z_l = M^{j_l}$. We shall also need the expression for I_j itself; it is

$$I_j = i^{-n} M^{j(n-k-1)} \int_0^\infty dr\, r^{k-n+2} f(M^{-2j} r^2)^m\, f(M^{-2j-2} r^2)^{n-m} \times \int_S d\omega \int_0^{2\pi} d\varphi\, e^{-in\varphi} \int_0^1 dt \frac{\partial \phi}{\partial r}(tr, \varphi, \omega) \prod_{w=k+1}^{n-1} M^{-j} \theta_w(0, 0, \omega) \tag{2.111}$$

with ϕ given by (2.106).

2.7. Convergence of the Renormalized Green Functions

The power counting bounds show that divergences in the scale sums of graphs contributing to the Green functions come from unrenormalized two-legged insertions, as discussed at length in the Introduction. In this section we show that the renormalized Green functions converge in the limit $I \to -\infty$ in every order in perturbation theory, i.e., that the scale sum for the value of any graph converges. The bound for this value depends on the order of perturbation theory r, and for some graphs it contains a factor $r!$. We show that under the stated assumptions, in particular because of the nonnesting condition A3, these factorials in bounds for single graphs can arise only from the lack of decay of those forks f with $E(G_f^J) = 4$ for which the graph $\tilde{G}(f)$ is nonoverlapping. For the overlapping graphs, such factorials do not arise even if G_f^J is four-legged, because the improved power counting always produces enough decay to make the scale sum convergent instead of marginal. We state this precisely in this section and prove Theorems 1.2 and 1.3.

We shall show finiteness of the renormalized Green functions by deriving power counting bounds for the two- and four-legged effective vertices that arise in the scale flow. The two-legged vertices correspond to the r- and c-forks. Although dealing with these effective two- and four-legged vertices is a standard procedure of handling trees and labeled graphs, what is not standard here is the behavior of the c-forks. Normally[2,3] they are constants, and therefore any derivative acting on them gives zero (such derivatives always arise from the Taylor expansions used to perform renormalization cancellations). In the nonspherical case, however, the c-forks are still momentum dependent because the shape of S is not fixed by a symmetry. Since the scales of c-forks are summed downward, the ordinary power counting bounds are insufficient to show convergence of a differentiated c-fork, and the improved power counting bounds are necessary.

For $M > 1$, $n \in \mathbb{N}$, $h \in \mathbb{Z}$, and $\varepsilon > 0$, define the function

$$\lambda_n(h, \varepsilon) = \sum_{p=1}^{\infty} (|h| + p + 1)^n M^{-\varepsilon p} \tag{2.112}$$

Obviously, λ is monotonically increasing in $|h|$ and $(|h| + 1)^m \lambda_n(h, \varepsilon) \leqslant \lambda_{m+n}(h, \varepsilon)$. This function bounds the effect of n of the marginal four-forks mentioned above on the scale sum of the fork below this. The following properties allow one to collect the accumulated effect of such factors when summing scales down a fixed tree.

Lemma 2.44. Let $\varepsilon > 0$ and $M_0(\varepsilon) = 2^{2/\varepsilon}$. Then for all $M \geqslant M_0$, $a \geqslant \varepsilon$, all $m, n \in \mathbb{Z}$, and all $j \in \mathbb{Z}$, $j < 0$:

(i) $$\lambda_m(j,\varepsilon)\,\lambda_n(j,\varepsilon) \leqslant \lambda_{m+n}(j,\varepsilon)$$

(ii) $$\sum_{l\leqslant j} (|l|+1)^m\, M^{al}\lambda_n(l,\varepsilon/2) \leqslant (1-M^{-\varepsilon/2})^{-1}\,\lambda_{m+n}(j,\varepsilon/2)\,M^{aj} \leqslant 2\lambda_{m+n}(j,\varepsilon/2)\,M^{aj} \tag{2.113}$$

(iii) $$\sum_{h=j+1}^{0} (|h|+1)^m\, M^{h\varepsilon/2}\lambda_n(h,\varepsilon/2) \leqslant 2\lambda_{m+n}(j,\varepsilon/2) \tag{2.114}$$

(iv) $$\sum_{h=j+1}^{0} (|h|+1)^m\, M^{-a(h-j)}\lambda_n(h,\varepsilon/2) \leqslant \lambda_{m+n}(j,\varepsilon/2) \tag{2.115}$$

(v) At fixed n,

$$\lambda_n\left(k,\frac{\varepsilon}{2}\right) \leqslant a_n k^n + b_n \tag{2.116}$$

with

$$a_n = \frac{2^n}{M^\varepsilon - 1}, \qquad b_n = \sum_{p\geqslant 1} (2p+1)^n\, M^{-\varepsilon p} \tag{2.117}$$

Proof. (i) By definition,

$$\begin{aligned}
\lambda_m(j,\varepsilon)\,\lambda_n(j,\varepsilon) &= \sum_{p,q=1}^{\infty} (|j|+p+1)^m\,(|j|+q+1)^n\, M^{-\varepsilon(p+q)} \\
&\leqslant \sum_{p,q=1}^{\infty} (|j|+\max\{p,q\}+1)^{m+n}\, M^{-\varepsilon(p+q)} \\
&\leqslant 2\sum_{\mu=1}^{\infty} (|j|+\mu+1)^{m+n} \sum_{\nu=1}^{\mu} M^{-\varepsilon(\mu+\nu)} \\
&\leqslant 2\sum_{\nu=1}^{\infty} M^{-\varepsilon\nu}\lambda_{m+n}(j,\varepsilon) \\
&\leqslant 2\,\frac{M^{-\varepsilon}}{1-M^{-\varepsilon}}\,\lambda_{m+n}(j,\varepsilon)
\end{aligned} \tag{2.118}$$

Since $M^\varepsilon \geqslant 4$, (i) holds.

(ii) Setting $l=j-k$, $k\geqslant 0$, we can rewrite the sum as

$$M^{aj}\sum_{k=0}^{\infty}(|j|+k+1)^m M^{-ak}\sum_{p=0}^{\infty}M^{-\varepsilon p/2}(|j|+k+p+1)^n$$
$$\leqslant M^{aj}\sum_{q=0}^{\infty}M^{-\varepsilon q/2}(|j|+q+1)^n\sum_{k=0}^{q}M^{-\varepsilon k/2}(|j|+k+1)^m \tag{2.119}$$

In the sum over k, we estimate each term by $|j|+k+1\leqslant|j|+q+1$. Extending the sum over k to ∞, we obtain the result.

(iii) Since for each h in the sum $|h|\leqslant|j|$,

$$(|h|+1)^m\lambda_n(h,\varepsilon/2)\leqslant(|j|+1)^m\lambda_n(j,\varepsilon/2)\leqslant\lambda_{n+m}(j,\varepsilon/2)$$

we have

$$\sum_{h=j+1}^{0}(|h|+1)^m M^{h\varepsilon/2}\lambda_n(h,\varepsilon/2)\leqslant\lambda_{n+m}(j,\varepsilon/2)\sum_{h\leqslant 0}M^{h\varepsilon/2}$$
$$\leqslant 2\lambda_{n+m}(j,\varepsilon/2) \tag{2.120}$$

(iv) As in the proof of (iii),

$$\sum_{h=j+1}^{0}(|h|+1)^m M^{-a(h-j)}\lambda_n\left(h,\frac{\varepsilon}{2}\right)$$
$$\leqslant\lambda_{n+m}\left(j,\frac{\varepsilon}{2}\right)\sum_{h\geqslant j+1}M^{-a(h-j)}$$
$$\leqslant\frac{M^{-a}}{1-M^{-a}}\lambda_{n+m}\left(j,\frac{\varepsilon}{2}\right)$$
$$\leqslant\frac{1}{(2^{2/\varepsilon})^{\varepsilon}-1}\lambda_{n+m}\left(j,\frac{\varepsilon}{2}\right)\leqslant\lambda_{n+m}\left(j,\frac{\varepsilon}{2}\right) \tag{2.121}$$

(v)

$$\lambda_n\left(k,\frac{\varepsilon}{2}\right)=\sum_{p=1}^{\infty}(k+p+1)^n M^{-\varepsilon p}$$
$$=\sum_{p=1}^{k-1}(k+p+1)^n M^{-\varepsilon p}+\sum_{p\geqslant k}(k+p+1)^n M^{-\varepsilon p}$$
$$\leqslant(2k)^n\sum_{p=1}^{\infty}M^{-\varepsilon p}+\sum_{p\geqslant 1}(2p+1)^n M^{-\varepsilon p}$$
$$\leqslant a_n k^n+b_n \tag{2.122}$$

with a_n and b_n as given in the statement of the lemma. ∎

Remark 2.45. Given any labeled graph G^J with tree t contributing to the renormalized Green functions, we will now construct the quotient graph G' mentioned in Remark 2.16(ii) and the corresponding tree t'. We recall that G' is to have the following properties: G' has only two- and four-legged vertices, with vertex functions that are either interaction vertices or values of 1PI two- or four-legged subgraphs. The only nontrivial two-legged subdiagrams of G' that correspond to forks of t' are strings of two-legged vertices. Any nontrivial four-legged subdiagram of G' that corresponds to a fork of t' consists of a single four-legged vertex with strings of two-legged vertices appended. The significance of this in the inductive proof of finiteness of the infrared limit is that the scale sum over the scales of forks $f \in t'$ can be easily bounded once the vertex functions of G' are controlled—and the latter will be covered by an appropriate inductive hypothesis because they are of lower order.

Let ϕ be the root of t, and let $f_1,..., f_r$ be all forks of t that satisfy: for all $k \in \{1,..., r\}$, the number of external legs of G_{f_k} is two or four, and f_k is minimal in the sense that there is no fork g such that $\phi < g < f_k$ and G_g has two or four external legs. Let $\tilde{t}$ be the tree rooted at ϕ and obtained from t by trimming t at $f_1,..., f_r$ (i.e., by collapsing t_{f_i}, as defined in Definition 2.27, to a leaf) so that $f_1,..., f_r$ are leaves of $\tilde{t}$, with vertex functions $Val(G^J_{f_k})$. The result is a graph $\tilde{G}$ and a tree $\tilde{t}$, such that no fork of $\tilde{t}$ corresponds to a nontrivial two- or four-legged subdiagram. $\tilde{t}$ is not yet the tree with the stated properties because the G_{f_k} need not be 1PI. When this is the case, we extend the tree further above f_k to construct t'.

Let f be one of the forks $f_1,..., f_r$. If G_f is 1PI, f is a leaf of t'. If G_f is 1P reducible, then in the transition from $\tilde{t}$ to t', f is replaced by one fork with some leaves above it. We now specify the procedure for getting t' in the different possible cases.

If f is a c-fork with $E(G_f) = 2$, G_f must be 1PI, since $l\ Val(G) = 0$ for any 1P reducible graph by the support properties of the propagator C_j.

If f is an r-fork with $E(G_f) = 2$ and G_f is 1P reducible, let $\mathscr{C}$ be the set of lines $l \in L(G_f)$ such that G_f disconnects if l is cut. If all lines in $\mathscr{C}$ are cut, what remains of G_f falls into s connected components θ_i. By the definition of $\mathscr{C}$, all the θ_i are two-legged graphs. Moreover, they are all 1PI. Thus G_f is a string of two-legged 1PI subdiagrams $\theta_1,..., \theta_s$ joined by the lines in $\mathscr{C}$, and

$$Val(G^J_f)(p) = S(p) = \left(\prod_{i=1}^{s-1} T_i(p)\, C_{j_i}(p)\right) T_s(p) \tag{2.123}$$

where $T_k = \mathscr{P}_{\theta_k}\ Val(\theta_k)$. Note that the external lines of G^J_f must have scales $j_{\pi(f)}$ or below, while each line of $\mathscr{C}$ must have scale j_f or above. By momentum conservation, the scale assignments, and (2.16), all $l \in \mathscr{C}$ must carry scales $j_l = j_f = j_{\pi(f)} + 1$. Since $\ell C_j = 0$, ℓ applied to the value of such a string is zero, so effectively $1 - \ell$ is replaced by 1. Let θ be one of $\theta_1, \ldots, \theta_s$. Then θ can be $\theta = G_g$, where g is an r- or c-fork directly above f, i.e., $\pi(g) = f$ and $\mathscr{P}_0 = 1 - \ell$ or ℓ, or θ is a two-legged graph of root scale j_f, in which case $\mathscr{P}_0 = 1$. Let us call this latter case a same scale insertion. We continue the construction of t' by reinstalling the fork f and adding, for every $k \in \{1, \ldots, s\}$, a leaf b_k above f which has vertex function T_k. Now, G_f just consists of the lines of $\mathscr{C}$ and the vertices $b_1, \ldots, b_s$.

If G_f is four-legged and 1PI, f is a leaf of t'.

Finally, if G_f is four-legged and 1PR, remove the strings attached to G_f according to Remark 2.23, and add a leaf, above the fork f, for the 1PI core of G_f, as well as for each 1PI two-legged subdiagram θ_i of the strings. The strings have the same properties as the ones discussed in the 1P-reducible r-fork case.

Doing this for all of $f_1, \ldots, f_r$, we obtain the tree t'. By construction, $G' = \tilde{G}(t')$ has the desired properties.

Finally, we note that if G is 1PI, G' is as well, since it is a quotient graph of G.

The relation between the scale sums for G and G' is

$$\sum_{J \in \mathscr{J}(t,j)} Val(G^J) = \sum_{J_1 \in \mathscr{J}(t',j)} Val(G'^{J_1}) \tag{2.124}$$

In this formula, $\mathscr{J}$ is as usual, but the vertices w of G' carry a scale index j_w, as discussed in Remark 2.16. If $j_w = 0$, w is also a vertex of G, and the associated vertex function is $\hat{v}$. Otherwise, j_w is the root scale of a subgraph of G whose value is a vertex function in G' [given by (2.75)] and j_w is summed over. For fixed $j_{\pi(w)}$, the summed vertex function is

$$F_w = \mathscr{P}_w \sum_{j_w} \sum_{J \in \mathscr{J}(t_w, j_w)} Val(\tilde{G}(t_w)) \tag{2.125}$$

where $\mathscr{P}_w \in \{1 - \ell, \ell\}$ for two-legged vertices associated to forks, and $\mathscr{P}_w = 1$ for two-legged vertices corresponding to same scale insertions and for four-legged vertices. The range of summation for j_w is: a sum $I \leqslant j_w \leqslant j_{\pi(w)}$ for a c-fork, a sum $j_{\pi(w)} + 1 \leqslant j_w < 0$ for an r-fork or a four-legged vertex, and no sum at all, but $j_w = j_{f = \pi(w)}$, for a same scale insertion. The last point is important because these diagrams do not have $1 - \ell$ in front, but the correct factor M^{j_f} is there because their scale is fixed. For a fork $f \in t$, let

$$n_f = |\{f' \in t\colon f' > f,\ \tilde{G}(f') \text{ nonoverlapping},\ E(G_{f'}) = 4,\ G_{f'} \text{ 1PI}\}| \tag{2.126}$$

n_f indeed depends only on G and t, but not on the scale assignment $J \in \mathscr{J}(t, j)$.

Theorem 2.46. Let G be a graph with $E(G) = 2m$ external legs and t be a tree rooted at a fork ϕ compatible with G, so that (t, G) contributes to the renormalized effective action at scale j, $G^I_{j, 2m, r}$ (see Remark 2.16). For $I < j < 0$ and $J \in \mathscr{J}(t, j)$, let $Val(G^J)$ denote the value of the labeled graph G^J with root scale $j_\phi = j$. Let ε be the volume improvement exponent of Proposition 1.1. Let $|\cdot|_s$ be as in (1.44) and (1.45); recall that for $2m$-point functions with $m > 1$, the supremum is taken over all $2m - 1$ independent external momenta entering into G. The numbers of vertices and of internal lines of G are denoted by $|V(G)|$ and $|L(G)|$, respectively.

(i) Let G be 1PI. There is a constant Q_0 such that for $s \in \{0, 1, 2\}$

$$\sum_{J \in \mathscr{J}(t, j)} |Val(G^J)|_s \leqslant Q_0^{|L(G)|} \, |\hat{v}|_s^{|V(G)|} \, \lambda_{n_\phi}\left(j, \frac{\varepsilon}{2}\right) M^{jY_s} \tag{2.127}$$

where

$$Y_s(G) = \begin{cases} (1 + \varepsilon - s) & \text{if} \quad E(G) = 2m = 2 \\ 2 - m - s & \text{if} \quad E(G) \geqslant 4 \text{ and } \tilde{G}(\phi) \text{ is nonoverlapping} \\ 2 - m - s + \varepsilon & \text{if} \quad E(G) \geqslant 4 \text{ and } \tilde{G}(\phi) \text{ is overlapping} \end{cases} \tag{2.128}$$

(ii) Let $X = 1 + W_1 + W_2$, where W_s is as in Lemma 2.3(iii), let K_0 and K_1 be as in Lemma 2.35, U_2 as in Lemma 2.42, and

$$K_2 = \max\left\{2(2 + d\,|\mathbf{P}|_1), \frac{2\sqrt{2}\, M^2}{u_0}, M^{2(1+\varepsilon)}, M^4 U_2\right\} \tag{2.129}$$

Then

$$Q_0 = \frac{18 d K_0 K_1 K_2 X^2}{1 - M^{-1}} \tag{2.130}$$

will do.

(iii) For $s \leqslant 1$ and $E(G) = 2m \geqslant 4$, the estimate (i) also holds for one-particle reducible graphs.

(iv) As $I \to -\infty$, $\sum_{J \in \mathscr{J}(t, j)} Val(G^J)$ converges in $|\cdot|_1$ to a function that obeys the bound (i), and, for G two-legged and 1PI, $\sum_{j=I}^{-1} \sum_{J \in \mathscr{J}(t, j)} Val(G^J)$ converges in $|\cdot|_1$.

Proof. We take (i)–(iv) as induction hypotheses and do induction over the depth of the pair (t, G), which is defined as

$$P = \max\{k: \exists f_1 > f_2 > \cdots > f_k > \phi \\ \text{with } E(G^J_{f_i}) \in \{2, 4\} \text{ for } 1 \leqslant i \leqslant k\} \tag{2.131}$$

In other words, given any leaf of the tree t, there are at most P two-legged or four-legged forks on the unique path between the root ϕ of the tree and this leaf. Let $\mathcal{N}$ be as in Lemma 2.31 and recall that $\phi \in \mathcal{N} \Leftrightarrow \tilde{G}(\phi)$ non-overlapping. Also, call $Q(G) = Q_0^{|L(G)|}$.

If $P = 0$, G has no two- or four-legged subgraphs associated to forks of t, so $n_\phi = 0$. Since no G_f is two-legged, $\mathcal{N} = \emptyset$ or $\mathcal{N} = \{\phi\}$, depending on whether $\tilde{G}(\phi)$ is overlapping or not. Also, once (i)–(iii) are proven, (iv) is trivial, since $Val(G^J)$ does not depend on I at all for $P = 0$. We note right away that the only places where I will enter for $P > 0$ are in the values of two-legged subdiagrams through the lower limit of the scale sum for c-forks.

Case 1. $P = 0$, $s = 0$ with $E(G) \geqslant 4$ or $E(G) = 2$ and $\tilde{G}(\phi)$ overlapping. By Theorem 2.40,

$$\sum_{J \in \mathcal{J}(t,j)} |Val(G^J)|_0 \\ \leqslant (4K_0)^{|L(G)|} K_1 M^{\varepsilon j\, 1(\phi \notin \mathcal{N})} M^{\varepsilon j^*(\phi)\, 1(\phi \in \mathcal{N})} \\ \times M^{D_\phi j} \sum_{J \in \mathcal{J}(t,j)} \prod_{f > \phi} M^{D_f(j_f - j_{\pi(f)})} \prod_{V_4(G)} |\hat{v}|_0 \tag{2.132}$$

[see (2.24) for the definition of the D_f]. Since there are no two- or four-legged forks (except possibly ϕ), $D_f \leqslant -1$ holds for all $f > \phi$. In the sum over $J \in \mathcal{J}(t, j)$, j_f runs from $j_{\pi(f)}$ to -1, since there are no c-forks $f > \phi$ (the corresponding subgraph would be two-legged). Thus every scale sum is bounded by

$$\sum_{j_f > j_{\pi(f)}} M^{D_f(j_f - j_{\pi(f)})} \leqslant \sum_{k \geqslant 0} M^{-k} \leqslant \frac{1}{1 - M^{-1}} \tag{2.133}$$

Doing the scale sums downward from the leaves of t in the standard way, we get a factor $(1 - M^{-1})^{-1}$ for every fork of t, except for ϕ. Since every fork f corresponds to a subgraph of G, the number of forks is bounded by $|L(G)|$. Thus

$$\sum_{J \in \mathcal{J}(t,j)} |Val(G^J)|_0 \leqslant \left(\frac{4K_0K_1}{1 - M^{-1}}\right)^{|L(G)|} M^{D_\phi j} M^{\varepsilon j\, 1(\phi \notin \mathcal{N})} |\hat{v}|_0^{|V(G)|} \tag{2.134}$$

Recalling that $D_\phi = 2 - m$ if G has $2m$ external legs, and that $\phi \notin \mathcal{N} \Leftrightarrow \tilde{G}(\phi)$ overlapping, we obtain the statement for $s = 0$.

Case 2. $P = 0$, $s \in \{1, 2\}$ with $E(G) \geqslant 4$ or $E(G) = 2$ and $\tilde{G}(\phi)$ overlapping. Now we apply $s \leqslant 2$ derivatives with respect to the external momenta. The derivative can act on vertices (interaction lines) or on fermion lines in the spanning tree of the graph. A bound for the number of targets for each derivative is thus $2\,|V(G)| - 1$. Because G is connected, $|V(G)| \leqslant |L(G)| + 1$. If the derivatives act on interaction lines, their effect can be bounded by $|\hat{v}|_s$. By Lemma 2.3(iii), the effect of s derivatives acting on fermion lines can be bounded by an additional factor $W_s M^{-sj_1}$, where j_1 is the lowest scale at which the derivative acts. Moreover, the value of the differentiated graph can be bounded using Theorem 2.40, since all support properties remain the same as before.

If $\tilde{G}(\phi)$ is overlapping, we use $M^{-j_1} \leqslant M^{-j}$ to bound

$$\begin{aligned}\sum_{J \in \mathcal{J}(t,j)} |Val(G^J)|_s &\leqslant K_1 (4K_0)^{|L(G)|} \left[(2\,|L(G)| + 1)\, X\right]^2 |\hat{v}|_s^{|V(G)|} \\ &\quad \times M^{(D_\phi + \varepsilon - s) j} \sum_{J \in \mathcal{J}(t,j)} \prod_{f > \phi} M^{D_f (j_f - j_{\pi(f)})} \\ &\leqslant Q(G)\, |\hat{v}|_s^{|V(G)|} M^{(D_\phi + \varepsilon - s) j} \lambda_0(j, \varepsilon) \end{aligned} \tag{2.135}$$

as before. If $\tilde{G}(\phi)$ is nonoverlapping and $E(G) \geqslant 4$, a similar bound holds without the ε. So far irreducibility has played no role, so (iii) holds for $P = 0$ even with $s = 2$.

Case 3. $P = 0$, $E(G) = 2$, and $\tilde{G}(\phi)$ nonoverlapping, $s \in \{1, 2\}$. Now $\tilde{G}(\tau_\phi)$ is a nonoverlapping two-legged graph. It is 1PI because it is a quotient graph of the 1PI graph G. Let $s \geqslant 1$. By Remark 2.41, the derivative does not act on lines of $\tilde{G}(\tau_\phi)$, but only on lines with scale $\geqslant j^*(\phi)$, where $j^*(\phi)$ is the lowest scale above τ_ϕ, as in Theorem 2.40, so its effect can be bounded by a factor $X^s M^{-sj^*(\phi)}$. If $\tau_\phi = t$, then $j^*(\phi) = 0$, and the derivative can act only on the interaction lines or lines of scale zero. Otherwise, by Theorem 2.40, we have a factor $M^{\varepsilon j^*(\phi)}$. Since $s \geqslant 1 \geqslant \varepsilon$ and $0 \geqslant j^*(\phi) \geqslant j$,

$$M^{-(s-\varepsilon) j^*(\phi)} \leqslant M^{(\varepsilon - s) j} \tag{2.136}$$

and we again obtain the bound (2.135). Note that if the derivative acts only on interaction lines, the bound is true since $1 \leqslant M^{(\varepsilon - s) j}$.

Case 4. $P = 0$, $E(G) = 2$, and $\tilde{G}(\phi)$ nonoverlapping, $s = 0$. We use the representation (2.92) for $Val(G^J)$ and Lemma 2.42. Pick a string S_1 that contains a line of scale j. This is possible because $\tilde{G}(\phi)$ is nonoverlapping. Recall that j is the root scale, hence the lowest possible scale for any line of the graph. Let

$$g(p_1, q)_{a_1\alpha_{m_1}\beta\beta'} = \sum_{(\alpha_i)_{i\notin\{1,m_1\}}} \int \prod_{i=2}^{m_1-1} (d^{d+1}p_i\,(S_i(p_i))_{\alpha_{m_1-1+i}\alpha_i}) \times (\mathscr{U}_{v_1})_{\alpha_1\cdots\alpha_{m_1-1}\beta\alpha_{m_1}\cdots\alpha_{2m_1-2}\beta'}(p_1,..., p_{m_1-1}, q, p_1,..., p_{m_1-1}) \tag{2.137}$$

Then

$$V_j = (Val(G^J))_{\beta\beta'}(q) = \sum_{\alpha,\alpha'} \int d^{d+1}p\,(S_1(p))_{\alpha\alpha'}\, g(p, q)_{\alpha'\alpha\beta\beta'} \tag{2.138}$$

The string S_1 can contain only insertions at scale j, i.e., vertices with generalized self-contractions of scale j because $P=0$. So

$$S_1(p) = C_j(p_0, e(\mathbf{p}))^m\, C_{j+1}(p_0, e(\mathbf{p}))^{n-m} \prod_{w=1}^{n-1} T_w(p) \tag{2.139}$$

with $m \geqslant 1$, and where the T_w are values of 1PI two-legged subdiagrams with root scale precisely j, and which are nonoverlapping on scale j (those are not excluded by $P=0$ because we did not use normal ordering). In the notation of Lemma 2.42,

$$V_j = \int_0^\infty r\,dr \int_0^{2\pi} d\varphi\,(ire^{i\varphi})^{-n} f(M^{-2j}r^2)^m f(M^{-2j-2}r^2)^{n-m} \times \int_S d\omega\,(\phi(r\cos\varphi, r\sin\varphi, \omega) - \phi(0, 0, \omega)) \tag{2.140}$$

with

$$\phi(p_0, \rho, \omega) = J(\rho, \omega)\, g(p_0, \mathbf{p}(\rho, \omega), q) \prod_{w=1}^{n-1} T_w(p_0, \mathbf{p}(\rho, \omega)) \tag{2.141}$$

[as in Lemma 2.42, $\int d\varphi\, e^{-in\varphi}\phi(0, 0, \omega) = 0$ because $\phi(0, 0, \omega)$ does not depend on φ]. By Taylor expansion,

$$V_j = \int_0^\infty r^2\,dr \int_0^{2\pi} d\varphi\,(ire^{i\varphi})^{-n} f(M^{-2j}r^2)^m f(M^{-2j-2}r^2)^{n-m} \times \int_S d\omega \int_0^1 dt\,(\cos\varphi\,\partial_0 + \sin\varphi\,\partial_1)\,\phi(tr\cos\varphi, tr\sin\varphi, \omega)) \tag{2.142}$$

As in the proof of Lemma 2.42, the extra factor of r gained by Taylor expansion alone would improve the scale behavior by a factor M^j. However, there are now derivatives acting on either J, or g, or one of the T_w's. We consider all these cases separately.

If the derivative acts on J, we use Lemma 2.1(iv) to bound $|J|_1$. Moreover, we can use the inductive hypothesis (IH) (proven as case 1) for the four-legged graph F whose value is g to bound

$$|g|_0 \leqslant Q(F)\, |\hat{v}|_0^{|V(F)|} \tag{2.143}$$

Thus this contribution to V_j is

$$\begin{aligned} &\leqslant |J|_1\, |g|_0 \prod_{w=1}^{n-1} |T_w|_0 \int_0^\infty dr\, r^{2-n} f(M^{-2j} r^2) \int_S d\omega \\ &\leqslant M^{2j} M^{2n}\, |J|_1\, Q(F)\, |\hat{v}|_0^{|V(F)|} \prod_{w=1}^{n-1} (|T_w|_0\, M^{-j}) \end{aligned} \tag{2.144}$$

The root scale behavior of the two-legged graphs is (applying the IH to their external vertex and the power counting bounds for the propagators) $Q_w\, |\hat{v}|_0^{|V(T_w)|}\, M^j$, so the statement follows for this term.

If the derivative acts on one of the T_w's, it can act only on an interaction line, or on a scale where the two-legged graph overlaps. Thus, bounding its value by the IH (proven as case 2 or 3), we have

$$|T_w|_1 \leqslant M^{\varepsilon j} Q_w\, |\hat{v}|_1^{|V(T_w)|} \tag{2.145}$$

so the contribution from this term is bounded by

$$\leqslant M^{j(1+\varepsilon)} M^{2n}\, |J|_0\, Q(F)\, |\hat{v}|_0^{|V(F)|} \prod_{w=1}^{n-1} (Q_w\, |\hat{v}|_0^{|V(T_w)|}) \tag{2.146}$$

If the derivative acts on g, it affects only $\mathscr{U}_{v_1}$. There, it can hit any line of scale j^* or higher, where j^* is the scale at which G overlaps, or an interaction line. We now wish to use the argument of Theorem 2.40 to extract the volume gain. The crucial step in this argument, as applied to the current situation, is bounding the two overlapping momentum loop integrals

$$\begin{aligned} Y = \sup_{q, \phi, t} \int_{\mathscr{B}} d^d \mathbf{p}_1 \int_0^\infty dr \int_S d\omega\, f(M^{-2j} r^2)\, 1(|e(\mathbf{p}_1)| < M^{j^*}) \\ \times 1(|e(v_1 \mathbf{p}_1 + v_2 \mathbf{p}(tr \sin\phi, \omega) + q))| < M^{j^*}) \end{aligned} \tag{2.147}$$

Here $\mathbf{p}_1$ is the spatial momentum of a loop of $\mathscr{U}_{v_1}$, the two factors $1(\cdots)$ come from the cutoffs on two lines of that loop, the integrals over r and

ω come from the momentum integral (2.142) for the string S_1, and the factor $f(M^{-2j}r^2)$ comes from the cutoff of one of the lines of S_1. This two-loop integral is not quite of the form of Proposition 1.1, with the most serious difference being the appearance of $\mathbf{p}(tr\sin\phi, \omega)$ in place of $\mathbf{p}(r, \omega)$.

However, writing $\mathbf{p}_1 = \mathbf{p}(\rho_1, \omega_1)$, doing the same Taylor expansion as at the beginning of Appendix A, and using that in the support of the integrand, $r \leqslant M^j \leqslant M^{j^*}$, we get

$$\begin{aligned} &1(|e(v_1\mathbf{p}_1 + v_2\mathbf{p}(tr\sin\phi, \omega) + q))| \leqslant M^{j^*}) \\ &\quad \leqslant 1\left(|e(v_1\mathbf{p}(0,\omega_1) + v_2\mathbf{p}(0,\omega_2) + \mathbf{q})| \leqslant \left(1 + 2\frac{|e|_1}{u_0}\right) M^{j^*}\right) \end{aligned} \tag{2.148}$$

so that

$$\begin{aligned} Y &\leqslant \sup_{q,\phi,t} \int_{-M^{j^*}}^{M^{j^*}} d\rho_1 \int_0^{M^j} dr \int_S d\omega_1\, |J(\rho_1, \omega_1)| \int_S d\omega \\ &\quad \times 1\left(|e(v_1\mathbf{p}(0,\omega_1) + v_2\mathbf{p}(0,\omega_2) + \mathbf{q})| \leqslant \left(1 + 2\frac{|e|_1}{u_0}\right) M^{j^*}\right) \\ &\leqslant 2\,|J|_0\, M^j M^{j^*} W\left(\left(1 + 2\frac{|e|_1}{u_0}\right) M^{j^*}\right) \end{aligned} \tag{2.149}$$

with W the function defined in Appendix A. So by Lemma A.1, the integral is bounded by $C_{\text{vol}} M^j M^{j^*} M^{\varepsilon j^*}$. Substituting this into the proof of Theorem 2.40, we find that the term in which the derivative acts on g is bounded by

$$\begin{aligned} &\leqslant M^{\varepsilon j^*} M^{-j^*} W_1 K_1 \left(\frac{4K_0}{1 - M^{-1}}\right)^{|L(F)|} |\hat{v}|_1^{|V(F)|} M^{2n} M^{(3-n)j} \prod_{w=1}^{n-1} |T_w|_0 \\ &\leqslant M^{(1+\varepsilon)j} M^{2n} W_1 K_1 \left(\frac{4K_0}{1 - M^{-1}}\right)^{|L(F)|} |\hat{v}|_1^{|V(F)|} \prod_{w=1}^{n-1} (Q_w |\hat{v}|_0^{|V(T_w)|}) \end{aligned} \tag{2.150}$$

Collecting the constants into the U_s mentioned in Lemma 2.42, this proves the statement for case 4.

Now assume $P \geqslant 1$ and (i)–(iv) to be proven for all $P' < P$. Construct the graph G' and its tree t' as in Remark 2.45. Recall that, by construction, G' has only two- and four-legged vertices. Furthermore, any two-legged subgraph corresponding to a fork must be a string of two-legged vertices and any four-legged subgraph corresponding to a fork must consist of a single four-legged vertex with some strings of two-legged vertices appended. Recall the definition of the vertex functions F_w and the scale sums involved therein from Remark 2.45. By construction of G', all graphs $\tilde{G}(t_w)$ whose values V_w appear in the definition of F_w are 1PI and two- or four-legged, and they are of depth at most $P-1$, so the inductive hypothesis applies to them. Our procedure is to estimate the norms of $|F_w|_s$ for $s \in \{0, 1\}$ (and, when F_w is four-legged, for $s=2$) first, using the inductive hypothesis, and then to apply this to complete the induction step using the case $P=0$, since, by construction, G' has depth zero. We abbreviate

$$Q_w = Q(\tilde{G}(t_w))\ |\hat{v}|_2^{|V(\tilde{G}(t_w))|}$$

and call $n_w = n_f$ if w comes from the fork $f \in t$.

Let F_w belong to a c-fork. Then

$$|F_w|_s = \left| \ell \sum_{j_w = I}^{j_{\pi(w)}} V_w \right|_s \tag{2.151}$$

For $s=0$, by $|\ell T|_0 \leqslant |T|_0$, the inductive hypothesis (IH), and Lemma 2.44(ii),

$$\begin{aligned} |F_w|_0 &\leqslant Q_w \sum_{j_w = I}^{j_{\pi(w)}} \lambda_{n_w}\left(j_w, \frac{\varepsilon}{2}\right) M^{j_w Y_0} \\ &\leqslant 2Q_w \lambda_{n_w}\left(j_{\pi(w)}, \frac{\varepsilon}{2}\right) M^{j_{\pi(w)}(1+\varepsilon)} \end{aligned} \tag{2.152}$$

If $s=1$, we use $|\ell T|_1 \leqslant (1 + d\,|P|_1)\,|T|_1$, where d is the spatial dimension, P is the projection onto S, and $|P|_1 = \max_i |P_i|_1$, then

$$\begin{aligned} |F_w|_1 &\leqslant (1 + d\,|P|_1) \sum_{j_w = I}^{j_{\pi(w)}} |V_w|_1 \\ &\leqslant Q_w (1 + d\,|P|_1) \sum_{j_w = I}^{j_{\pi(w)}} \lambda_{n_w}\left(j_w, \frac{\varepsilon}{2}\right) M^{j_w Y_1} \\ &\leqslant 2Q_w (1 + d\,|P|_1)\, M^{\varepsilon j_{\pi(w)}} \lambda_{n_w}\left(j_{\pi(w)}, \frac{\varepsilon}{2}\right) \end{aligned} \tag{2.153}$$

by Lemma 2.44(ii), since $Y_1 = \varepsilon$.

If F_w belongs to an r-fork, then

$$|F_w|_s = \left| (1-\ell) \sum_{j_w > j_{\pi(w)}} V_w \right|_s \tag{2.154}$$

For $s=0$, by (2.37), and because the momentum p flowing through $\tilde{G}_w$ must be in supp $C_{j_{\pi(w)}}$, i.e., $|ip_0 - e(\mathbf{p})| \leqslant M^{j_{\pi(w)}+2}$,

$$
\begin{aligned}
|F_w|_0 &\leqslant \sum_{j_w > j_{\pi(w)}} M^{j_{\pi(w)}+2} \frac{\sqrt{2}}{u_0} |V_w|_1 \\
&\leqslant Q_w \frac{\sqrt{2}\, M^2}{u_0} M^{j_{\pi(w)}} \sum_{j_w > j_{\pi(w)}} \lambda_{n_w}\left(j_w, \frac{\varepsilon}{2}\right) M^{j_w \varepsilon} \\
&\leqslant 2Q_w \frac{\sqrt{2}\, M^2}{u_0} M^{j_{\pi(w)}} \lambda_{n_w}\left(j_{\pi(w)}, \frac{\varepsilon}{2}\right)
\end{aligned}
\tag{2.155}
$$

by Lemma 2.44(iii).

For $s=1$, we ignore the renormalization gain, and bound

$$
|(1-\ell)\, V_w|_1 \leqslant (2 + d\, |\mathbf{P}|_1)\, |V_w|_1 \tag{2.156}
$$

Inserting the IH, the scale sum is as in the $s=0$ case, and

$$
|F_w|_1 \leqslant (2 + d\, |\mathbf{P}|_1)\, 2Q_w \lambda_{n_w}\left(j_{\pi(w)}, \frac{\varepsilon}{2}\right) \tag{2.157}
$$

If F_w belongs to a same-scale insertion, $j_w = j_{\pi(w)}$, and so

$$
|F_w|_s \leqslant Q_w \lambda_{n_w}\left(j_{\pi(w)}, \frac{\varepsilon}{2}\right) M^{j_{\pi(w)} Y_s(\tilde{G}(t_w))} \tag{2.158}
$$

for all $s \leqslant 2$ follows directly from the IH.

If F_w belongs to a four-legged fork of t, the IH implies

$$
|F_w|_s \leqslant Q_w \sum_{j_w > j_{\pi(w)}} \lambda_{n_w}\left(j_w, \frac{\varepsilon}{2}\right) M^{j_w Y_s(\tilde{G}(t_w))} \tag{2.159}
$$

Bound

$$
M^{j_w Y_s} \leqslant M^{-s j_{\pi(w)}} M^{Y_0 j_w}
$$

If $\tilde{G}(w)$ is overlapping, $Y_0 = \varepsilon$, so by Lemma 2.44(iii),

$$
|F_w|_s \leqslant 2Q_w \lambda_{n_w}\left(j_{\pi(w)}, \frac{\varepsilon}{2}\right) M^{-s j_{\pi(w)}} \tag{2.160}
$$

If $\tilde{G}(w)$ is nonoverlapping, $Y_0 = 0$, and the scale sum grows logarithmically, i.e., as $|j_{\pi(w)}|$, and

$$|F_w|_s \leqslant Q_w \lambda_{n_w+1}\left(j_{\pi(w)}, \frac{\varepsilon}{2}\right) M^{-sj_{\pi(w)}} \tag{2.161}$$

In summary we have for $s \leqslant 1$ the bounds

$$|F_w|_s \leqslant Q_w K_2 M^{j_{\pi(w)}(1-s)} \lambda_{n_w}\left(j_{\pi(w)}, \frac{\varepsilon}{2}\right) \tag{2.162}$$

for the vertex functions of two-legged vertices w of G', and for all $s \leqslant 2$ the bounds

$$|F_w|_s \leqslant 2Q_w M^{-sj_{\pi(w)}} \lambda_{\tilde{n}_w}\left(j_{\pi(w)}, \frac{\varepsilon}{2}\right) \tag{2.163}$$

for all four-legged vertices w of G', with

$$\tilde{n}_w = \begin{cases} n_w & \text{if} \quad \tilde{G}(w) \text{ is overlapping} \\ n_w + 1 & \text{if} \quad \tilde{G}(w) \text{ is nonoverlapping} \end{cases} \tag{2.164}$$

We return to G' and complete the inductive step. Choose a spanning tree T' as in Lemma 2.35(i) for G' and fix the momenta, to obtain $Val(G'^J)$ in the form (2.83), with the $\mathscr{U}_w$ given by the F_w in the present case.

Case 5. $P > 0$, $E(G) \geqslant 2$, and $\tilde{G}(\phi)$ overlapping, $s \in \{0, 1\}$. Let q be an external momentum of G, and denote $\partial_\beta = \partial/\partial q_\beta$. Then

$$\begin{aligned} \partial_\beta^s \, Val(G'^J) = \sum_\sigma \sum_{\text{spins } \alpha} \int \prod_{l \in L(G') \backslash L(T')} d^{d+1} p_l \prod_{l \in L(G')} \partial_\beta^{\sigma_l} C_{j_l}((p_l)_0, e(\mathbf{p}_l))_{A_l} \\ \times \prod_{w \in V_4(G')} \partial_\beta^{\sigma_w} F_w(p_1^{(w)}, p_2^{(w)}, p_3^{(w)})_{A_w} \prod_{w \in V_2(G')} \partial_\beta^{\sigma_w} F_w(p^{(w)})_{A_w} \end{aligned} \tag{2.165}$$

where the A denote the spin assignments for vertex functions and propagators, as defined in the graph rules. Here σ: $L(G') \cup V(G') \to \{0, 1\}$ keeps track of which factor gets differentiated, so exactly one of its components is nonzero, $\sum_l \sigma_l + \sum_w \sigma_w = 1$. To count the number of terms in the sum over σ, observe that $\partial_\beta C_{j_l} = 0$ if $l \notin T'$. Since $|L(T')| = V(G') - 1$, the sum over σ is bounded by $2\,|V(G')|$ times the maximum of the summand.

$\tilde{G}(\phi)$ is a quotient graph of G', so G' is overlapping. Applying Lemma 2.35(ii), using (2.162) and (2.163), and bounding $M^{-sj_{\pi(w)}} \leqslant M^{-sj}$, we obtain

$$\max_{\beta} |\partial_\beta \, Val(G'^J)|$$

$$\leqslant K_1(4K_0)^{|L(G')|} \, 2 \, |V(G')| \, X \prod_{w \in V(G')} Q_w$$

$$\times M^{j(\varepsilon + D_\phi(G') - s)} \prod_{\substack{f \in t' \\ f > \phi}} M^{D_f(G')(j_f - j_{\pi(f)})} \prod_{w \in V_2(G')} (K_2 M^{j_{\pi(w)}})$$

$$\times \prod_{w \in V_4(G')} \left(2\lambda_{\tilde{n}_w} \left(j_{\pi(w)}, \frac{\varepsilon}{2} \right) \right) \tag{2.166}$$

By (2.24), since G' has two- and four-legged vertices, $V(G'_f) = V_2(G'_f) + V_4(G'_f)$, so

$$D_f(G') = L(G') - 2(V(G') - 1) = \tfrac{1}{2}(4 - E(G'_f)) - V_2(G'_f) \tag{2.167}$$

By Lemma 2.44(i) and the definition of n_ϕ,

$$\prod_{w \in V_4(G')} \lambda_{\tilde{n}_w} \left(j_{\pi(w)}, \frac{\varepsilon}{2} \right) \leqslant \lambda_{n_\phi} \left(j, \frac{\varepsilon}{2} \right) \tag{2.168}$$

Using the telescope formula $(j_\phi = j)$

$$j_{\pi(w)} = j_\phi + \sum_{\substack{f \in t' \\ \phi < f \leqslant \pi(w)}} (j_f - j_{\pi(f)}) = j_\phi + \sum_{\substack{f \in t' \\ f > \phi}} (j_f - j_{\pi(f)}) \, 1(w \in G_f'^J) \tag{2.169}$$

we get

$$\sum_{w \in V_2(G')} j_{\pi(w)} = j_\phi \, |V_2(G')| + \sum_{\substack{f \in t' \\ f > \phi}} (j_f - j_{\pi(f)}) \sum_{w \in V_2(G')} 1(w \in G_f'^J) \tag{2.170}$$

so

$$\prod_{w \in V_2(G')} M^{j_{\pi(w)}} = M^{j_\phi \, |V_2(G')|} \prod_{\substack{f \in t' \\ f > \phi}} M^{(j_f - j_{\pi(f)}) \, |V_2(G_f'^J)|} \tag{2.171}$$

and we see that all $D_f(G')$ get renormalized to

$$D_f^{(\mathrm{R})}(G') = D_f(G') + V_2(G'_f) = \tfrac{1}{2}(4 - E(G'_f)) \tag{2.172}$$

Inserting these estimates into (2.166), summing over $J \in \mathscr{J}(t, j)$, and remembering (2.124), we find

$$\sum_{J \in \mathscr{J}(t,j)} |Val(G^J)|_s$$
$$\leqslant (2d\,|V(G')|\,X+1)\,K_1(4K_0)^{|L(G')|}\,K_2^{|V(G')|} \prod_{w \in V(G')} Q_w$$
$$\times M^{j(\varepsilon - s + D_\phi^{(\mathrm{R})}(G'))} \lambda_{n_\phi}\left(j, \frac{\varepsilon}{2}\right) \sum_{J \in \mathscr{J}(t',j)} \prod_{\substack{f \in t' \\ f > \phi}} M^{D_f^{(\mathrm{R})}(G')(j_f - j_{\pi(f)})} \tag{2.173}$$

We are now back to the case of zero depth, since, by construction, if f is a two-legged or four-legged fork for G', then $j_f = j_{\pi(f)} + 1$ by conservation of momentum. So there is no corresponding scale sum and the last scale sum is now identical to that of the case with zero depth. It produces a factor $(1 - M^{-1})^{-1}$ for all forks of t' with $E(G_f) > 4$, except for ϕ. The product of the various constants is now again bounded by $Q(G)$, since the number of lines of the subgraphs G_w and that of G' add up to $|L(G)|$. Finally, we note that $D_\phi^{(\mathrm{R})}(G') = \frac{1}{2}(4 - E(G')) = 2 - m$. This proves (i) and (ii) for $s \leqslant 1$ and also (iii) since we have used the assumption that G is 1PI only to bound $|V(G)| \leqslant |L(G)|$. For general connected graphs, $|V(G)| \leqslant |L(G)| + 1$, which only changes the constant. For $s = 2$ we will need the 1PI assumption since we cannot afford to have two derivatives acting on c-forks.

Case 6. $P > 0$, $E(G) \geqslant 2$, and $\tilde{G}(\phi)$ overlapping, $s = 2$. For $s = 2$, we have to apply another momentum derivative to (2.165). It can act again on at most $2V(G')$ targets. However, we have to avoid having two derivatives act on an F_w coming from a c-fork because the scale sum down to I would diverge as $I \to -\infty$ in that case. Whenever the second derivative acts on the same two-legged vertex as the first (no matter whether this vertex comes from a c- or an r-fork), we remove it by integration by parts as follows. Since G is 1PI, so is G', so the momentum through any two-legged vertex is a linear combination of momenta, at least one of which is a loop momentum p. So we can rewrite the derivative

$$\frac{\partial}{\partial q_\gamma} T(p \pm q) = \pm \frac{\partial}{\partial p_\gamma} T(p \pm q) \tag{2.174}$$

Integrating by parts with respect to p distributes the derivative on at most $|V| - 1$ other lines and at most V vertices. Using (2.162) and (2.163), the estimate follows as in the case $s \leqslant 1$, but with the constant $(2\,|V|\,X)$ replaced by $(2\,|V|\,X)^2$ because there are more terms in the sum, and because the derivatives can now act on two C's.

Case 7. $P > 0$, $E(G) \geqslant 4$, and $\tilde{G}(\phi)$ nonoverlapping. Just delete all the $M^{\varepsilon j}$ from cases 5 and 6.

Case 8. $P > 0$, $E(G) = 2$, and $\tilde{G}(\phi)$ nonoverlapping, $s = 0$. We proceed as in the case $P = 0$. The value of G again takes the form of (2.137)–(2.139), but with the T_{w} replaced by 1PI insertions with values F_{w} [see (2.125)] on the strings S_i of $G(t_\phi)$, where F_{w} may now belong to a c-fork, an r-fork, or an SSI. If F_{w} belongs to a c-fork, the additional $M^{\varepsilon j}$ can be read off (2.152). Since the scale of an SSI is fixed, the additional $M^{\varepsilon j}$ follows directly from the IH. Thus if any c-fork or SSI is on the string, the statement follows immediately.

There remains the case where all insertions on the string are r-forks, i.e., $\mathscr{P}_{w} = 1 - \ell$ in (2.125). Then the value of G is given by (2.111), with $k = n - 1$ (only r-forks), and with ϕ and Δ_{w} given by (2.106) and (2.103). The derivative $\partial/\partial r$ acting on ϕ in (2.111) can act on J, or on γ, or on the Δ_{w}, similarly to case 4 above.

If the derivative acts on J, $|J|_1 \leqslant A_1/u_0^2$, and by Theorem 2.46(i) (proven as cases 5 and 6 above),

$$|\gamma|_0 \leqslant |g|_0 \leqslant Q(F)\,|v|_2^{|V(F)|}\, M^{\varepsilon/2} \lambda_{n_f}\left(j, \frac{\varepsilon}{2}\right) \tag{2.175}$$

with F the four-legged graph whose value is g. The insertions to which the Δ_{w} belong are of depth $\leqslant P - 1$, so by the IH and (2.103), $|\Delta_{w}|_0$ is bounded. Thus this contribution is bounded by $Q(G)\,|v|_2^{|V(G)|}\,\lambda_{n_\phi}(j, \varepsilon/2)\, M^{2j}$.

The case of the derivative acting on $\gamma(r, \varphi, \omega) = g(r\cos\varphi, \mathbf{p}(r\sin\varphi, \omega))$ is completely similar to that given in case 4.

If the derivative acts on one of the Δ_{w}, we apply the IH for $s = 2$ to the two-legged graph of depth $\leqslant P - 1$ that produces Δ_{w}. The gain of $M^{\varepsilon j}$ follows immediately from the IH. This completes the induction step for case 8.

Case 9. $P > 0$, $E(G) = 2$, and $\tilde{G}(\phi)$ nonoverlapping, $s \in \{1, 2\}$. If $s = 1$, we choose the spanning tree constructed in Lemma 2.38; then the derivative with respect to the external momentum can act only on the vertex function $\mathscr{U}_{v_1}$. Since the volume improvement is at the same scale, the statement follows even without an application of Lemma 2.42. If $s = 2$, at most one derivative can act below $j^*(\phi)$, and that happens only if it has to be rerouted to avoid having two derivatives act on a single c-fork. Bounding it by M^{-j} and then proceeding as in the case $s = 1$, we arrive at a similar bound.

Finally, (iv). The effective vertices in G' have vertex functions that depend on I, and that converge as $I \to -\infty$ by the IH. Since the scales of r-forks and four-legged diagrams are summed over a region that does not depend on I, and since a same-scale insertion has no scale sum at all, (iv) will be proven if we can show that the scale sum for a c-fork, which runs from I to $j_{\pi(f)}$, also converges as $I \to -\infty$. Let $\mathscr{C}$ be the Banach space $(C^1([-1,1]\times\mathscr{B},\mathbb{C}), |\cdot|_1)$. The sequence $g^I = (g_n^I)_{n\leqslant 0}$ given by

$$g_n^I = \begin{cases} \sum_{J\in\mathscr{J}(t_w,n)} Val(G_w^J) & \text{if} \quad n \in \{I,\ldots,j_{\pi(w)}\} \\ 0 & \text{otherwise} \end{cases} \tag{2.176}$$

is an element of the space $\ell^1(\mathbb{Z}_-, \mathscr{C})$ by (i). By the IH applied to G_w, there is $(g_n)_{n\leqslant 0}$ such that $g_n^I \to g_n$ in $|\cdot|_1$ as $I \to -\infty$; in other words, $g^I \to g$ pointwise as a sequence. Let $f \in \ell^1(\mathbb{Z}_-, \mathscr{C})$ be the sequence $f_n = M^{\varepsilon n} Q(G_w)\, \lambda_{n_w}(n, \varepsilon/2)$; then

$$\|g^I\|_{\ell^1(\mathbb{Z}_-,\mathscr{C})} \leqslant \|f\|_{\ell^1(\mathbb{Z}_-)} \tag{2.177}$$

for all $I < 0$. By dominated convergence, $g \in \ell^1(\mathbb{Z}_-, \mathscr{C})$ and

$$\|g^I - g\|_{\ell^1(\mathbb{Z}_-,\mathscr{C})} = \sum_{n\leqslant 0} |g_n^I - g_n|_1 \to 0 \tag{2.178}$$

as $I \to -\infty$, so $\sum_{n\in\{I,\ldots,j_{\pi(w)}\}} F_n$ converges in $|\cdot|_1$, which shows that $\gamma_j^I = \sum_{J\in\mathscr{J}(t,j)} Val(G^J)$ converges in $|\cdot|_1$ as $I \to -\infty$. Morever, if G itself is two-legged and 1PI,

$$|\gamma_j^I| \leqslant \varphi_j = M^{\varepsilon j} Q(G)\, \lambda_{n_\phi}(j, \varepsilon/2) \tag{2.179}$$

Now repeat the dominated convergence argument for the γ_j^I to see that $\sum_{j\geqslant I} \gamma_j^I$ also converges as $I \to -\infty$. ∎

Theorem 2.46 contains the most important information, that of the renormalization flow of the two- and four-legged, i.e., relevant and marginal, effective vertices. For overlapping four-legged graphs, the bounds show that the scale behavior is not marginal, but irrelevant in the usual language of the renormalization group. The convergence as $I \to -\infty$ allows us to view the flow of effective actions to the uncutoff limit $G_{j,2m,r} = \lim_{I\to-\infty} G_{j,2m,r}^I$.

Theorem 2.47. Let $|\cdot|'$ be as in (1.46), n_ϕ as in Remark 2.45, and let (t, G) be fixed. Let

$$\|\cdot\| = \begin{cases} |\cdot|_1 & \text{if} \quad E(G) = 2 \text{ and } G \text{ is 1PI} \\ |\cdot|_0 & \text{if} \quad E(G) = 2 \text{ and } G \text{ is 1P-reducible,} \\ & \qquad \text{or } E(G) = 4 \text{ and } \tilde{G}(\phi) \text{ is overlapping} \\ |\cdot|' & \text{otherwise} \end{cases} \tag{2.180}$$

Then

$$V_I(t, G) = \sum_{j=I}^{-1} \sum_{J \in \mathcal{J}(t,j)} Val(G^J)$$

converges in $\|\cdot\|$ and satisfies

$$\|\lim_{I \to -\infty} V_I(t, G)\| \leqslant n_\phi! \cdot \text{const}^{|L(G)|} \tag{2.181}$$

Proof. For $E(G) = 2$, or $E(G) = 4$ and $\tilde{G}(\phi)$ overlapping, the statement follows from Theorem 2.46 by summation over j, noting that for $\alpha > 0$,

$$\sum_{j<0} M^{\alpha j} \lambda_n(j, \varepsilon/2) \leqslant 2\lambda_n(0, \varepsilon/2) \leqslant \text{const}'' \cdot n! \tag{2.182}$$

It remains to show the bound in $|\cdot|'$. Construct $\tilde{G}$ and $\tilde{t}$ as in Remark 2.45. The two-legged vertices of $\tilde{G}$ are either strings of vertices of G' or c-forks. By Theorem 2.46, the scale behavior is

$$T_w \leqslant \text{const}^{L_w} \cdot \lambda_{n_w}(j_{\pi(w)}, \varepsilon/2)\, M^{j_{\pi(w)}}$$

for two-legged vertices and

$$F_w \leqslant \text{const}^{L_w} \cdot \lambda_{\tilde{n}_w}(j_{\pi(w)}, \varepsilon/2)$$

with $\tilde{n}_w = n_w$ if $\tilde{G}(w)$ is overlapping and $\tilde{n}_w = n_w + 1$ if it is nonoverlapping. Inserting the second part of Lemma 2.4 and Lemma 2.44(i), it follows now as in refs. 2 and 3 that the scale sums $\sum_{J \in \mathcal{J}(t,j)} Val(G^J)$ converge as $I \to \infty$, and that they are uniformly (in I) bounded by a summable function in j. The convergence as $I \to -\infty$ now follows by imitating the proof of Theorem 2.46(iv), using $L^1(([-1, 1] \times \mathscr{B})^{n-1} \times \{\uparrow, \downarrow\}^n, \mathbb{C})$ instead of $\mathscr{C}$. ∎

Remark 2.48. The convergence statements of Theorem 1.2 follow directly from this, recalling that the graphs contributing to Σ and K are 1PI and two-legged, and that the sum over trees t at fixed G is always finite. Under the hypotheses of Theorem 1.3, $n_\phi = 0$ for all t that are compatible with G, so, taking into account the $1/n_f!$ to bound the sum over trees by const′, Theorem 1.3 also follows. The local Borel summability bound requires an adapted induction scheme that combines the summation over trees with the bounds of Theorem 2.46, using Felder's Lemma. We will not repeat the proof here; it is similar to the one given in ref. 2.

Remark 2.49. Theorem 2.47 states that the value of every four-legged graph that is overlapping on root scale converges in the sup norm to a continuous function. The only four-legged graphs that may produce a singularity in the four-point function are thus the nonoverlapping four-legged graphs. By Lemma 2.26, these are the ladder diagrams. The "dangerous divergences" mentioned in many places in the literature are those of the four-point function. Their "danger" is that they can produce factorial growth of the value of individual diagrams when they appear as subdiagrams and thus may prevent convergence of the renormalized expansion in λ (even though every order is now finite). Theorem 2.47 shows that for our class of models with a nonnested Fermi surface, these "dangerous divergences" can only be produced by dressed ladder diagrams, so that it suffices to investigate them to see whether r factorials in the value of individual diagrams of order r appear.

3. THE DERIVATIVE WITH RESPECT TO THE BAND STRUCTURE

Let D_h be the directional derivative with respect to e, as defined in (1.51). It is obvious from the formula for the value of graphs and the way e appears in the propagators and in the projection that for a fixed infrared cutoff $I > -\infty$, all Green functions have bounded D_h, and, moreover, their multiple derivatives with respect to e exist as multilinear operators. However, the norms of these operators diverge as $I \to -\infty$. In this section we show that $D_h K_r^I(e)$ converges as $I \to -\infty$, and that $K_r = \lim_{I \to -\infty} K_r^I$ is differentiable in e in the sense of Fréchet. To get bounds that are suitable for removal of the cutoff I, we have to rearrange some contributions that appear divergent at first.

To motivate why there is a problem taking this derivative, we first explain how it affects power counting. Abbreviating $f(M^{-2j}(p_0^2 + e(\mathbf{p})^2)) = f_j(p)$, we have that the derivative

$$D_h C_j(p_0, e(\mathbf{p})) = \left(\frac{f_j(p)}{(ip_0 - e(\mathbf{p}))^2} + \frac{2M^{-2j} e(\mathbf{p})\, f_j'(p)}{ip_0 - e(\mathbf{p})} \right) h(\mathbf{p}) \tag{3.1}$$

obeys

$$|D_h C_j(p_0, e(\mathbf{p}))| \leqslant \text{const} \cdot M^{-2j}\, 1(|ip_0 - e(\mathbf{p})| \in [M^{j-2}, M^j])\, |h|_0 \tag{3.2}$$

which is a factor M^{-j} worse than the usual scaling behavior of C_j [Lemma 2.3(iii)]. By power counting, a two-legged graph on scale j behaves as

M^j, so D_h removes the decay and seems to make the scale sum marginally divergent. Similarly, D_h also acts on the projection ℓ and can upset renormalization cancellations.

In brief, taking a derivative of $1/(ip_0 - e(\mathbf{p}))$ effectively produces a square of the denominator, which, as discussed in the Introduction, is not locally integrable. The problem that the infrared singularity gets stronger when derivatives are applied also appears, for example, in Euclidean field theory with propagators singular at zero momentum, when differentiating with respect to a mass in the infrared.

However, the singularity of the propagator is on a surface in our case and this makes a big difference under the nonnestedness assumption A3. The improved power counting estimate implies that for contributions from graphs that are overlapping on root scale, the scale sum is actually still convergent because of the volume improvement factor $M^{\varepsilon j}$, so that for these graphs, $\sum_J D_h \, Val(G^J)$ converges. For contributions from nonoverlapping graphs, one has to apply an integration by parts similar to Lemma 2.42 to show that the scale sum still converges. So the derivative of K is convergent without any further insertions or counterterms, because of the geometry of the singularity. The two above observations will be used to treat general labeled graphs using Lemma 2.31. Note that in contrast to derivatives with respect to the external momentum, where momentum routing implied that lines in the nonoverlapping parts of the graph are never hit by such a derivative, derivatives with respect to e will affect all lines, and it requires a separate argument to remove D_h from lines in the nonoverlapping parts of the tree. When taking norms, there will be several subtleties which we discuss in detail below.

3.1. Integration by Parts

We start with the simple observation that if $F \in C^1(\mathbb{R}, \mathbb{C})$, then $F \circ e$: $\mathscr{B} \to \mathbb{C}$ satisfies

$$D_h F(e(\mathbf{p})) = F'(e(\mathbf{p})) \, h(\mathbf{p}) \tag{3.3}$$

and

$$\nabla F(e(\mathbf{p})) = F'(e(\mathbf{p})) \, \nabla e(\mathbf{p}) \tag{3.4}$$

Thus, choosing δ as in Lemma 2.1(iii), for $\mathbf{p} \in \mathscr{U}_\delta(S)$,

$$D_h(F \circ e) = \left(\frac{h}{\mathscr{D}_u e} \, (\mathscr{D}_u F) \circ e \right) \tag{3.5}$$

where $\mathscr{D}_u = u \cdot \nabla$, as in Lemma 2.7. Since supp $C_j \subset \mathscr{U}_\delta(S)$ for all $j \leqslant -1$,

$$D_h C_j(p_0, e(\mathbf{p})) = \frac{h(\mathbf{p})}{\mathscr{D}_u e(\mathbf{p})} (\mathscr{D}_u C_j)(p_0, e(\mathbf{p})) \tag{3.6}$$

and this rewriting introduces no singularities since $|u \cdot \nabla e(\mathbf{p})| \geqslant u_0$. Obviously, then, for $X \in C^1([-1, 1] \times \mathscr{B}, \mathbb{C})$,

$$\int d^d\mathbf{p}\, X(\mathbf{p})\, D_h C_j(p_0, e(\mathbf{p}))$$

$$= -\int d^d\mathbf{p}\, C_j(p_0, e(\mathbf{p}))\, \nabla \cdot \left(\frac{hu}{\mathscr{D}_u e} X\right)(\mathbf{p}) \tag{3.7}$$

and thus

$$\int_{\mathscr{B}} D_h(C_j X) = -\int_{\mathscr{B}} C_j X \nabla \cdot \left(\frac{hu}{\mathscr{D}_u e}\right) + \int_{\mathscr{B}} C_j \delta_h X \tag{3.8}$$

where

$$\delta_h X = D_h X - \frac{h}{\mathscr{D}_u e} \mathscr{D}_u X \tag{3.9}$$

By the definition of δ_h, $\delta_h F = 0$ for all $F(p) = b(p_0, e(\mathbf{p}))$ and all h. In particular, $\delta_h C_j(p) = 0$. Note, however, that for $F(p) = C_j(p_0 + q_0, e(\mathbf{p} + \mathbf{q}))$, $\delta_h F$ will not be zero if $q \neq 0$. We shall only use integration by parts for nonoverlapping graphs, so by Remark 2.41 such shifts by additional momenta q will not occur. Moreover, X will be given by (2.93), so, to get the integration-by-parts formula, we only have to give the derivative of ℓ.

Lemma 3.1. Let u be fixed independently of e, so that $D_h u = 0$. Then

$$\begin{aligned} D_h \mathbf{P}(q) &= -\frac{h(\mathbf{P}(q))}{\mathscr{D}_u e(\mathbf{P}(q))} u(\mathbf{P}(q)) = -\ell \left(\frac{hu}{\mathscr{D}_u e}\right)(q) \\ D_h(\ell T) &= \ell \delta_h T \end{aligned} \tag{3.10}$$

On $U_\delta(S)$, $\delta_h \ell = \ell \delta_h$ and thus $\delta_h(1 - \ell) = (1 - \ell)\, \delta_h$.

Proof. Fix $\mathbf{q} \in \mathscr{U}_{\delta_0}(S)$. Changing e to $e + \alpha h$ moves the Fermi surface. The new surface is $\tilde{S} = \{\mathbf{p}: (e + \alpha h)(\mathbf{p}) = 0\}$. h is bounded, so for α small enough, $\tilde{S} \subset \mathscr{U}_\delta(S)$. By assumption, changing e does not change the curve γ used to define $\mathbf{P}(\mathbf{q})$, since γ is an integral curve determined by u and $\mathbf{q}$.

What changes is the intersection points of γ with $\tilde{S}$. Since this point moves on γ, and γ is an integral curve of u,

$$D_h \mathbf{P}(\mathbf{q}) = \beta u(\mathbf{P}(\mathbf{q})) \tag{3.11}$$

with a function $\beta = \beta(e, h, u, \mathbf{q})$. Inserting this into the equation

$$0 = D_h(e(\mathbf{P}(\mathbf{q})) = h(\mathbf{P}(\mathbf{q})) + \nabla e(\mathbf{P}(\mathbf{q})) \cdot D_h \mathbf{P}(\mathbf{q}) \tag{3.12}$$

we obtain $\beta = -\ell(h/\mathscr{D}_u e)$ and thus the statement for $\mathbf{P}$.

Since $(\ell T)(q) = T(0, \mathbf{P}(\mathbf{q}))$,

$$\begin{aligned} D_h \ell T(q) &= (D_h T)(0, \mathbf{P}(\mathbf{q})) + \nabla T(0, \mathbf{P}(\mathbf{q})) \cdot D_h \mathbf{P}(\mathbf{q}) \\ &= (\ell D_h T + \ell\, \nabla T \cdot D_h \mathbf{P})(q) \end{aligned} \tag{3.13}$$

which, by the formula for $D_h P$, implies (3.10). On $U_\delta(S)$, $\mathscr{D}_u \ell = 0$ by Lemma 2.7. Thus for $\mathbf{q} \in U_{\delta_0}(S)$,

$$(\delta_h \ell T)(q) = (D_h \ell T)(q) = (\ell \delta_h T)(q) \tag{3.14}$$

The statement for $1 - \ell$ is obvious. ▌

Remark 3.2. Allowing u to vary with e would have given an additional term parallel to the surface. Since that term vanishes linearly on S, it can be included without any problems, but the resulting expressions are more complicated. For our purposes, keeping u fixed is enough. Indeed, for the bound without ∇ acting on h, it is necessary since the additional term contains ∇h.

To use this for strings of two-legged insertions, we write the string of (2.93) as $S_i(p) = C_{j_{i,1}}(p) \cdot Y_i(p)$ with

$$Y_i = \prod_{k=1}^{w_i} (\mathscr{P}_{i,k} T_{i,k})\, C_{j_{i,k+1}} \tag{3.15}$$

and note that for $j_{i,k} \leqslant -1$, the momentum $\mathbf{p} \in \mathscr{U}_{\delta_0}(S)$, so that Lemma 3.1 applies. Since $\delta_h C_{j_{i,k}} = 0$ for all k,

$$\delta_h Y_i = \sum_{l=1}^{w_i} \prod_{k=1}^{w_i} (\mathscr{P}_{i,k} T'_{i,k,l})\, C_{j_{i,k+1}} \tag{3.16}$$

where

$$T'_{i,k,l} = \begin{cases} \delta_h T_{i,k} & \text{if } k = l \\ T_{i,k} & \text{if } k \neq l \end{cases} \tag{3.17}$$

We denote

$$S'_i(p) = C_{j_{i,1}}(p) \cdot \delta_h Y_i(p) \tag{3.18}$$

Lemma 3.3. Let G be a nonoverlapping, 1PI, two-legged graph as in Lemma 2.38 and v_1 its external vertex. Then

$$\begin{aligned} &D_h(Val(G^J)(C, \mathscr{U}_{v_1}, ..., \mathscr{U}_{v_n}))_{\beta\beta'}(q) \\ &\quad = (Val(G^J)(C, D_h \mathscr{U}_{v_1}, ..., \mathscr{U}_{v_n}))_{\beta\beta'}(q) \\ &\qquad + \sum_{l=1}^{n_1-1} \sum_{(\alpha_i)_i} \int \prod_{\substack{i=1 \\ i \neq l}}^{n_i-1} (d^{d+1}p_i\, (S_i(p_i))_{\alpha_{n_i+i}\alpha_i}) \\ &\qquad \times \left((S'_l(p_l))_{\alpha_{n_1+l}\alpha_l} - (S_l(p_l))_{\alpha_{n_1+l}\alpha_l} \nabla \cdot \left(\frac{hu}{\mathscr{D}_u e} \right)(p_l) - \frac{h}{\mathscr{D}_u e} u(p_l) \cdot \nabla_{p_l} \right) \\ &\qquad \times (\mathscr{U}_{v_1})_{\alpha_1 \cdots \alpha_{n_1-1}\beta\alpha_{n_1+1} \cdots \alpha_{2n_1-1}\beta'}(p_1, p_2, ..., p_{n_1-1}, q, p_1, p_2, ..., p_{2n_1-1}) \end{aligned} \tag{3.19}$$

Proof. For every string S_i attached to v_1, use (3.15) and integration by parts

$$\begin{aligned} &\int dp_l\, X(p_l)\, D_h S_l(p_l) \\ &\quad = \int dp_l\, X(p_l)\, \delta_h S_l(p_l) + \int dp_l\, X(p_l) \frac{h(p_l)}{(\mathscr{D}_u e)(p_l)} u(p_l) \cdot \nabla S_l(p_l) \\ &\quad = \int dp_l\, X(p_l)\, S'_l(p_l) - \int dp_l\, X(p_l)\, S_l(p_l)\, \nabla \cdot \frac{hu}{\mathscr{D}_u e} \\ &\qquad - \int dp_l\, S_l(p_l) \frac{hu}{\mathscr{D}_u e} \cdot \nabla X(p_l) \end{aligned}$$

to remove all derivatives from propagators. The momentum derivatives acting on $\mathscr{U}_{v_1}$, and the additional summands all arise from integration by parts. ∎

Remark 3.4. Less formally, one can say that after applying integration by parts, no derivative acts on a propagator of G, and that all derivatives act as δ_h on functions associated to higher forks in the tree, i.e., one of the $T_{i,k}$ in the strings gets differentiated in the terms in (3.19). If $T_{i,k} = T$ is again a nonoverlapping graph, it is a string of GST graphs; since

$\delta_h C_j = 0$, the derivative only affects 1PI subgraphs, which are then GST graphs, and we can again apply Remark 2.41 and Lemma 3.3. For the momentum derivative contained in δ, we will apply Theorem 2.46 directly. For the D_h contained in δ_h, we apply (3.19) again, to avoid having the derivative act on lines. This procedure can be iterated according to the recursive structure of the GST graph G, and all of $\mathscr{U}_{v_{i_1}}, ..., \mathscr{U}_{v_{i_r}}$ get differentiated in this procedure. This can be used to make all derivatives act on higher, overlapping parts of the tree, where the factors M^{-j} they produce are controlled by the improved power counting factors $M^{\varepsilon j}$. So, the upshot of (3.8) is that things can be arranged such that the derivative D_h also does not act on lines in the nonoverlapping part of $t(G^J)$ (as was the case for derivatives with respect to external momenta). However, because of the way integration by parts was done here, the price paid for this is that $|\nabla h|_0$, not only $|h|_0$, appears in the bound. The integration by parts is similar to the Taylor expansion in Lemma 2.42, which also produces ∇h terms when used on a string on which a factor of h from a $D_h C_j$ sits.

3.2. Bounds for the Directional Derivative

We now show convergence of the directional derivative and a bound that contains only $|h|_0$. Some parts of the proof will be subtle, and therefore we illustrate the two procedures for the lowest order contribution,

$$F(q) = \sum_{j=I}^{-1} F_j(q) = \sum_{j=I}^{-1} \int d^{d+1}p \; \hat{v}(q-p) \, C_j(p_0, e(\mathbf{p})) \tag{3.20}$$

We want to bound

$$D_h F(q) = \int d^{d+1}p \; \hat{v}(q-p) \sum_{j=I}^{-1} D_h C_j(p_0, e(\mathbf{p})) \tag{3.21}$$

To get the bound in $|h|_1$, we use (3.6) to write

$$D_h F_j(q) = \int d^{d+1}p \; \hat{v}(q-p) \frac{h(\mathbf{p})}{\mathscr{D}_u e(\mathbf{p})} (u \cdot \nabla) \, C_j(p_0, e(\mathbf{p})) \tag{3.22}$$

and integrate by parts, to get

$$D_h F_j(q) = -\int d^{d+1}p \; C_j(p_0, e(\mathbf{p})) \, \nabla \cdot \left(\hat{v}(q-p) \frac{h(\mathbf{p})}{\mathscr{D}_u e(\mathbf{p})} u(\mathbf{p}) \right) \tag{3.23}$$

and bound by (2.21), (2.22), and Lemma 2.1(ii)

$$|D_h F_j|_0 \leqslant K_0 M^j \cdot |\hat{v}|_1 \cdot |h|_1 \, |u|_1 \frac{1}{u_0}\left(1+\frac{|e|_2}{u_0}\right) \tag{3.24}$$

Thus, the scale sum converges absolutely if one allows a derivative to act on h, which gives

$$|D_h F|_0 \leqslant \text{const} \cdot |h|_1 \tag{3.25}$$

The bound in $|h|_0$ is obtained by an integration by parts in p_0, using

$$D_h C_j(p) = h(\mathbf{p})\left(i\frac{\partial}{\partial p_0} C_j(p_0, e(\mathbf{p})) - 2M^{-2j} f'(M^{-2j}(p_0^2 + e(\mathbf{p})^2))\right) \tag{3.26}$$

Then $D_h F_j = A_j + B_j$, where

$$\begin{aligned} A_j(q) &= \int đ^{d+1}p \, \hat{v}(q-p) \, i\partial_0 C_j(p_0, e(\mathbf{p})) \, h(\mathbf{p}) \\ &= -i \int đ^{d+1}p \, C_j(p_0, e(\mathbf{p}))(-\partial_0 \hat{v})(q-p) \, h(\mathbf{p}) \end{aligned} \tag{3.27}$$

(note that h does not get differentiated since it does not depend on p_0), so

$$|A_j(q)| \leqslant K_0 M^j \, |\hat{v}|_1 \, |h|_0 \tag{3.28}$$

and the scale sum $\sum_{j=I}^{-1} A_j$ converges absolutely. However,

$$B_j(q) = -2M^{-2j} \int đ^{d+1}p \, \hat{v}(q-p) \, f'(M^{-2j}(p_0^2 + e(\mathbf{p})^2)) \tag{3.29}$$

is $O(1)$, so $\sum_{j=I}^{-1} |B_j(q)| \geqslant \text{const} \cdot |I|$, and we have to perform the sum over j before taking $|\cdot|$ to get a sharper bound. We write

$$\sum_{j=I}^{-1} B_j(q) = -2 \int đ^{d+1}p \, \hat{v}(q-p) \, h(\mathbf{p}) \left[\frac{\partial}{\partial x} \sum_{j=I}^{-1} f(M^{-2j}x)\right]_{x = p_0^2 + e(\mathbf{p})^2} \tag{3.30}$$

By (2.13)

$$\sum_{j=I}^{-1} f(M^{-2j}x) = a(M^{-2I}x) - a(x) \tag{3.31}$$

and $a'(x) \neq 0$ only if $x \in [M^{-4}, M^{-2}]$ by (2.12). So

$$\left|\sum_{j=I}^{-1} B_j(q)\right| \leqslant 2\, |\hat{v}|_0\, |h|_0 \int d^{d+1}p \left(a'(p_0^2 + e(\mathbf{p})^2) + M^{-2I} a'\left(\frac{p_0^2 + e(\mathbf{p})^2}{M^{2I}}\right)\right)$$

$$\leqslant 2\, |\hat{v}|_0\, |h|_0\, |a'|_0 \int d^{d+1}p$$

$$\times (1(|ip_0 - e(\mathbf{p})| \leqslant 1) + M^{-2I}\, 1(|ip_0 - e(\mathbf{p})| \leqslant M^I))$$

$$\leqslant 4K_0\, |\hat{v}|_0\, |a'|_0\, |h|_0$$

$$\leqslant \text{const} \cdot |h|_0 \tag{3.32}$$

Thus, the divergence of $\sum |B_j|$ as $I \to -\infty$ is due to terms that depend on the scale decomposition. Once the partition of unity is resummed, all that remains is a boundary term at $j = I$ which is uniformly bounded as $I \to -\infty$. In general, the contributions to K where this procedure has to be applied are those from graphs that are nonoverlapping on root scale. There things are more complicated because the resummation of the partition of unity has to be done carefully, and because there are a lot more terms from the integration by parts. Note that the integration-by-parts formula (3.8) combines nicely with the ℓ-operations; also, the bound in $|h|_1$ avoids boundary terms and therefore allows us to show convergence, not only boundedness, of K' as $I \to -\infty$.

Theorem 3.5. Let G be a 1PI two-legged graph and t an associated tree so that (t, G) contributes to (2.76). Then there is $Q_1 > 0$ such that

$$\sum_{J \in \mathcal{J}(t,j)} |D_h\, Val(G^J)|_0 \leqslant Q_1^{|L(G)|} \lambda_{n_\phi}(j, \varepsilon/2)\, M^{\varepsilon j}\, |h|_1 \tag{3.33}$$

and for all $h \in C^1(\mathcal{B}, \mathbb{R})$

$$V_I(h, t, G) = \sum_{j \geqslant I} \sum_{J \in \mathcal{J}(t,j)} D_h\, Val(G^J) \quad \text{converges in } |\cdot|_0 \quad \text{as } I \to -\infty \tag{3.34}$$

Moreover, there is a constant const (depending on G, $\hat{v}$, and u but independent of $I < 0$) such that for all $h \in C^1(\mathcal{B}, \mathbb{R})$

$$\left|\sum_{j=I}^{-1} \sum_{t \sim G} \prod_{f \in t} \frac{1}{n_f!} \sum_{J \in \mathcal{J}(t,j)} D_h \ell\, Val(G^J)\right|_0 \leqslant \text{const} \cdot |h|_0 \tag{3.35}$$

Remark 3.6. Note that in (3.35), as in the above example, the norm is taken after summing over the root scale j. This, as well as the additional sum $\sum_{t \sim G}$ over trees associated to G, is necessary to resum the partition of unity properly. In all terms where the resummation of the partition of unity is not necessary, the sum over trees will be replaced by a maximum over trees times a constant, since the number of trees compatible with a fixed graph is always finite. The constant appearing in (3.35) depends on $\hat{v}$, e, u, and the graph G in the same way as in Theorem 2.46. In particular, it is uniform on the set $\mathscr{A}$ given in (1.53). To reduce notation a little, we are not going to trace the factors of λ through this proof, because it will be obvious in the proof that the factorials are again only produced by the nonoverlapping four-legged subgraphs. We denote a polynomial in $|j|$, whose coefficients may increase in inequalities and combine with other constants, by pol(j). In that notation, Theorem 2.46 reads

$$\left| \sum_{J \in \mathscr{J}(t,j)} Val(G^J) \right|_s \leqslant \text{pol}(j)\, M^{Y_s j} \tag{3.36}$$

for any $t \sim G$. We also assume $\varepsilon < 1$. Note also that (3.35) is not simply an application of Lemma 2.42, because the latter will cause ∇h terms in some cases.

Proof. By (3.10) and (3.9),

$$D_h \ell = \ell \left(D_h - \frac{h}{\mathscr{D}_u e} \mathscr{D}_u \right) \tag{3.37}$$

so the left side of (3.35) consists of two terms. Since $|\ell f|_0 \leqslant |f|_0$, the contribution to the second term from any fixed t, j is

$$\left| \sum_{J \in \mathscr{J}(t,j)} \ell \left(\frac{h}{\mathscr{D}_u e} u \cdot \nabla\, Val(G^J) \right) \right|_0 \leqslant \frac{|h|_0\, |u|_0}{u_0} \sum_{J \in \mathscr{J}(t,j)} |Val(G^J)|_1 \tag{3.38}$$

which, by Theorem 2.46(i), is

$$\leqslant \frac{|h|_0\, |u|_0}{u_0}\, Q_0^{|L(G)|} \lambda_{n_\phi} \left(j, \frac{\varepsilon}{2} \right) M^{\varepsilon j} \tag{3.39}$$

As

$$\frac{|h|_0\, |u|_0}{u_0} \sum_{j \leqslant -1} Q_0^{|L(G)|} \lambda_{n_\phi} \left(j, \frac{\varepsilon}{2} \right) M^{\varepsilon j} \leqslant \text{const} \cdot |h|_0$$

the second term is consistent with (3.33) and (3.35).

It remains to bound

$$\left| \sum_{j=I}^{-1} \sum_{t \sim G} \prod_{f \in t} \frac{1}{n_f!} \sum_{J \in \mathscr{J}(t,j)} \ell D_h \, Val(G^J) \right|_0 \tag{3.40}$$

Again, we do induction over the depth P of (t, G), defined in (2.131). The induction hypothesis (IH) is: if G is two-legged and 1PI and if (t, G) is of depth P, then (3.33) and (3.34) hold. Moreover:

(a) For all two-legged 1PI graphs G and all $i \in \{I, ..., -1\}$,

$$\left| \sum_{j=I}^{i} \sum_{\substack{t \sim G \\ \mathrm{depth}(t,G) \leqslant P}} \prod_{f \in t} \frac{1}{n_f!} \sum_{J \in \mathscr{J}(t,j)} D_h \, Val(G^J) \right|_0 \leqslant \mathrm{const} \cdot |h|_0 \tag{3.41}$$

Moreover, for any 1PI graph G with $E = 2$ or 4 external legs and for $s \in \{0, 1\}$,

$$\left| \sum_{J \in \mathscr{J}(t,j)} D_h \, Val(G^j) \right|_s \leqslant |h|_0 \, \mathrm{pol}(j) \, M^{Y'_s j} \tag{3.42}$$

(b) If $\tilde{G}(\phi)$ is overlapping, $Y'_s = 1 - (E/2) + \varepsilon - s$.

(c) If $\tilde{G}(\phi)$ is nonoverlapping,

$$Y'_s = \begin{cases} -1 - s & \text{if} \quad E = 4 \\ 0 & \text{if} \quad E = 2 \text{ and } s = 0 \\ \varepsilon - 1 & \text{if} \quad E = 2 \text{ and } s = 1 \end{cases} \tag{3.43}$$

The tree sum in (a) will be necessary to resum the scale decomposition in the way illustrated in the above example. An informal restatement of (b) and (c) is that the effect of D_h on the scale behavior is similar to that of a momentum derivative. Note that the h does not get differentiated even in the case $s = 1$; this hinges on the 1P irreducibility of G.

If $P = 0$, G has no nontrivial two- or four-legged forks (so G', constructed in Remark 2.45, is equal to G), there are no ℓ-operations, and therefore the only factors that depend on e are the propagators. By (3.1),

$$\begin{aligned} |D_h C_j(p)| &\leqslant M^{-j+2}(1 + 2 \, \|f'\|_\infty) \, M^{-j+2} \, |h|_0 \\ &\quad \times 1(|ip_0 - e(\mathbf{p})| \in [M^{j-2}, M^j]) \\ &\leqslant \mathrm{const} \cdot M^{-2j} \, 1_j(p) \, |h|_0 \end{aligned} \tag{3.44}$$

We shall use the just introduced notation $1_j(p)$ in what follows. Also, const will denote constants that may increase in inequalities and depend on $\hat{v}$, u, and the graph G, but not on j or the infrared cutoff I.

Case 1. $P=0$ and $\tilde{G}(\phi)$ overlapping. We use Theorem 2.40 to get

$$|D_h\, Val(G^J)|_0 \leqslant \text{const} \cdot |h|_0\, M^{j(2-m+\varepsilon)} \max_{l \in L(G)} M^{-j_l} \prod_{\substack{f \in t \\ f > \phi}} M^{D_f(j_f - j_{\pi(f)})} \tag{3.45}$$

Since $P=0$, $\max M^{-j_l} \leqslant M^{-j}$ for all $J \in \mathscr{J}(t, j)$, so

$$\sum_{J \in \mathscr{J}(t,j)} |D_h\, Val(G^J)|_0 \leqslant \text{const} \cdot |h|_0\, M^{(1-m+\varepsilon)j} \sum_{J \in \mathscr{J}(t,j)} \prod_{f > \phi} M^{D_f(j_f - j_{\pi(f)})} \tag{3.46}$$

and the scale sum over all $J \in \mathscr{J}(t, j)$ can be performed as in the $P=0$ case of the proof of Theorem 2.46. This proves (b) for $s=0$. If G is two-legged, the factor $M^{\varepsilon j}$ makes the scale sum over j convergent, and (3.33) (with $|h|_0$ instead of $|h|_1$) and (a) follow. Convergence as $I \to -\infty$ [that is, (3.34)] is now obvious because every summand is independent of I for $P=0$ and the series is absolutely convergent (recall also that the number of terms in the sum over $t \sim G$ is finite and independent of I). Similarly, an additional derivative with respect to the external momentum gives another factor $M^{-j_l} \leqslant M^{-j}$. Since G is 1PI, the spanning tree can always be chosen not to contain the line where h is, so that h does not get differentiated (alternatively, one can use integration by parts to remove the derivative from h). Thus

$$\sum_{J \in \mathscr{J}(t,j)} |D_h\, Val(G^J)|_1 \leqslant \text{const} \cdot M^{j(\varepsilon - m)}\, |h|_0 \tag{3.47}$$

which proves (b) for $s=1$.

Case 2. $P=0$, $E(G)=4$, and $\tilde{G}(\phi)$ nonoverlapping. The same bounds as above hold, with ε replaced by zero.

Case 3. $P=0$, $E(G)=2$, and $\tilde{G}(\phi)$ nonoverlapping. If $E=2$, $\tilde{G}(\phi)$ is an ST or GST diagram [see Definition 2.21(ii), (iii)] and so is $\tilde{G}(\tau_\phi)$, where τ_ϕ is as in Lemma 2.31. Thus $Val(\tilde{G}(\phi))$ takes the form given in Remark 2.41, with an effective vertex v_1 with $2m$ legs. We now consider the case in which all strings consist only of a single propagator ($w_i=0$). Insertions in these strings are treated as in $P>0$.

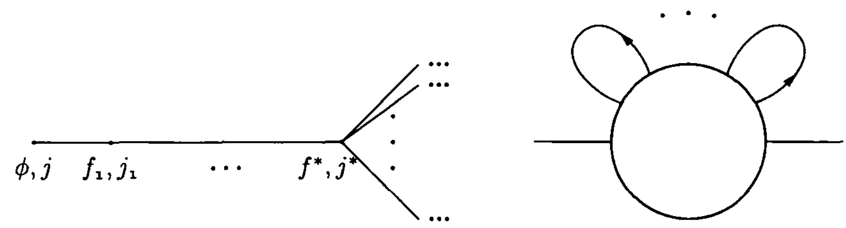

Fig. 19. The case $\tilde{G}(\phi)$ nonoverlapping, $P=0$.

If $2m=4$, v_1 is four-legged. Since $P=0$, v_1 is a vertex of scale zero. This can happen only in first order. Then

$$Val(G^J)(q) = Val(\tilde{G}(\phi))(q) = \int d^{d+1}p \, \{\hat{v}(q-p) \text{ or } \hat{v}(0)\} \, C_j(p_0, e(\mathbf{p})) \quad (3.48)$$

(3.33) follows from (3.24) and the bound (a) follows as in (3.32). Condition (3.34) is again obvious from (3.33). The bound (c) is obvious for $s=0$. For $s=1$, the derivative with respect to q acts only on $\hat{v}$, so (c) holds as well (actually, with a better exponent).

If $2m \geqslant 6$, G and t take the form shown in Fig. 19, with $n \geqslant 1$ and $f^* \in t$ the fork such that

$$\tilde{G}\begin{pmatrix} f^* \\ | \\ \tau_\phi \end{pmatrix}$$

is overlapping (f^* exists, since otherwise G would be nonoverlapping, hence a GST diagram, hence a ST diagram, since $P=0$, but then $2m=4$, since at scale zero there are only four-legged vertices), so

$$\sum_{J \in \mathscr{J}(t,j)} Val(G^J)(q)$$
$$= \sum_{j_1 > j} \sum_{j_2 > j_1} \cdots \sum_{j^* = j_n > j_{n-1}} \int \prod_{k=1}^{m^*} d^{d+1}p_k \, C_{i_k}((p_k)_0, e(\mathbf{p}_k)) \, W(q, p_1, ..., p_{m^*}) \quad (3.49)$$

Here

$$W = \sum_{J \in \mathscr{J}(t_{f^*}, j^*)} Val(\tilde{G}(t_{f^*})) \quad (3.50)$$

$\tilde{G}(t_{f^*})$ is a graph with $2m^*+2$ external legs, where $m^* \geqslant \max\{m-1, n-1\}$, and for each $k \in \{1,..., m^*\}$, either $i_k = j$ or there is $r \in \{1,..., n-1\}$ such that $i_k = j_r$. By assumption, $m \geqslant 3$, so there are at least two lines with

$i_k = j$. We may choose the labeling such that they are the lines for $k=1$ and $k=2$. Apply D_h to (3.49). If it acts on W, it can act only on a propagator of scale $j_l \geqslant j^*$, since $P=0$, and the net effect is, up to constants, a factor $M^{-j_l} \leqslant M^{-j^*}$ and a factor $h(\mathbf{p}_l)$. As mentioned in Remark 2.33, $\tilde{G}(f^*)$ need be neither overlapping nor 1PI, but

$$\tilde{G}\begin{pmatrix} f^* \\ | \\ \tau_\phi \end{pmatrix}$$

is overlapping and so the volume integral produces a factor $M^{\varepsilon j^*}$ by Theorem 2.40. Choosing the spanning tree as in Lemma 2.35, so that all p_k are loop lines, we have that Theorem 2.40 implies

$$\left| \int \prod_{k=1}^{m^*} d^{d+1}p_k \, C_{i_k}(p_k) \, D_h W(q, p_1, ..., p_{m^*}) \right| \leqslant \text{const} \cdot M^{i_1 + \cdots + i_{m^*}} M^{(\varepsilon - 1)j^*} M^{D_{f^*} j^*} \sum_{J \in \mathscr{J}(t_{f^*}, j^*)} \prod_{f > f^*} M^{D_f(j_f - j_{\pi(f)})} \tag{3.51}$$

Since $P=0$, $D_f < 0$ for all $f > f^*$, and so the scale sum over $J \in \mathscr{J}(t_{f^*}, j^*)$ converges by the argument of the $P=0$ case in the proof of Theorem 2.46. Since $\tilde{G}(f^*)$ has $2m^*+2$ external legs, $D_{f^*} = 2 - (m^*+1) = 1 - m^*$. Calling $m_r = \{k \in \{1,..., m^*\} : i_k = j_r\} \geqslant 1$, we have

$$m^* = \sum_{r=1}^{n} m_r + m - 1 \qquad \text{and} \qquad \sum_{k=1}^{m^*} i_k = \sum_{r=1}^{n-1} m_r j_r + (m-1) j \tag{3.52}$$

Inserting this and using again $M^{(\varepsilon-1)j^*} \leqslant M^{(\varepsilon-1)j}$, we have

$$\begin{aligned} &\left| \sum_{j^* \geqslant \cdots \geqslant j_1 \geqslant j} \int \prod_{k=1}^{m^*} d^{d+1}p_k \, C_{i_k}(p_k) \, D_h W(q, p_1, ..., p_{m^*}) \right| \\ &\quad \leqslant \text{const} \cdot |h|_0 \, M^{(\varepsilon-1)j} \sum_{j_{n-1} > \cdots > j_1 > j} M^{(m-1)j + \sum_{r=1}^{n-1} m_r j_r} \\ &\qquad \times \sum_{j^* > j_{n-1}} M^{(2-m)j^* - \sum_{r=1}^{n-1} m_r j^*} \\ &\quad \leqslant \text{const} \cdot |h|_0 \, M^{j\varepsilon} \sum_{j^* \geqslant j} M^{(m-2)(j-j^*)} \prod_{r=1}^{n-1} \sum_{i \leqslant j^*} M^{m_r(i-j^*)} \\ &\quad \leqslant \text{const} \cdot |h|_0 \, M^{j\varepsilon} \end{aligned} \tag{3.53}$$

This proves (c) for $s=0$, (3.33), and upon summation over j, (a) and (3.34), for the contribution to (3.40) where D_h acts on W. For this contribution, the only remaining case is (c) for $s=1$. With the spanning tree we chose, the derivative acts only on the functions associated to those lines in $D_h W$ in which the external momentum enters. When acting on a propagator, it produces a factor $\text{const} \cdot M^{-j_l} \leqslant \text{const} \cdot M^{-j}$. The only dangerous case is when h "sits" on the path through G_{f^*} through which the external momentum is routed, and the additional derivative acts on h. In that case we use integration by parts as in the proof of Theorem 2.46 to remove the derivative from h. This is possible because G is 1PI (note again that G_{f^*} need not be 1PI, and if it is not, the derivative will produce a factor M^{-j}, not just M^{-j^*}). Taking absolute values, we obtain the same bound as before, only multiplied by $\text{const} \cdot M^{-j}$. This proves (c) for $s=1$ for the term with D_h acting on W.

If D_h acts on one of the C_{i_k}, $k \in \{1,..., m^*\}$, we can assume without loss of generality that $k=1$, since $i_1 = i_2 = j$ and $i_k \geqslant j$ for all k, and the scale behavior degrades worst when the derivative occurs on the lowest scale. We have to bound

$$\left| \int d^{d+1}p_1 \, D_h C_j(p_1) \sum_{(j_k)} \int \prod_{r=2}^{m^*} d^{d+1}p_r \, C_{i_r}(p_r) \left(\frac{\partial}{\partial q_\alpha} \right)^s W(q, p_1,..., p_{m^*}) \right|_0 \tag{3.54}$$

(where $\sum_{(j_k)}$ is short for $\sum_{j_n > \cdots > j_1 > j}$) for $s=0$ and $s=1$. To see (3.33), we apply the integration-by-parts formula (3.7) to the integral over p_1. When the derivative acts on W, we get the bound (3.53). The term containing ∇h has the same scale behavior as if there had been no derivative at all. Condition (3.34) follows by the dominated convergence argument of the proof of Theorem 2.46(iv). To see (a) and (c), we use Lemma 2.42 in the p_2-integration to bound this by

$$\leqslant \int d^{d+1}p_1 \, |D_h C_j| \sum_{(j_k)} \int \prod_{r=3}^{m^*} d^{d+1}p_r \, |C_{i_r}(p_r)| \int d^{d+1}p_2 \, 1_j(p_2) \left| \left(\frac{\partial}{\partial q_\alpha} \right)^s W \right|_{1,j} \tag{3.55}$$

Estimating

$$|D_h C_j(p_0, e(\mathbf{p}))| \leqslant \text{const} \cdot M^{-2j} \, |h|_0 \, 1_j(p) \tag{3.56}$$

the C_{i_k} for $k \geqslant 3$ by Lemma 2.3(iii), and rearrranging the product, we find that (3.54) is bounded by

$$\leqslant \text{const} \cdot |h|_0 \sum_{(j_k)} \int \left(\prod_{r=1}^{m^*} d^{d+1}p_r \, M^{-i_r} \, 1_{i_r}(p_r) \right) \left| \left(\frac{\partial}{\partial q_\alpha} \right)^s W \right|_{1,j} \tag{3.57}$$

The derivative on W acts on a line in G_{f^*} and produces a factor const $\cdot M^{-j^*}$. By Theorem 2.40 (and since none of the lines $k = 1, \ldots, m^*$ is in the spanning tree), the result when $s = 0$ is

$$\leqslant \text{const} \cdot |h|_0 \sum_{(j_k)} M^{i_1 + \cdots + i_{m^*}} M^{(\varepsilon - 1) j^*} M^{D_{f^*} j^*} \sum_{J \in \mathcal{J}(t_{f^*}, j^*)} \prod_{f > f^*} M^{D_f (j_f - j_{\pi(f)})} \tag{3.58}$$

The sum over J is estimated as above, and also the rearrangement of the terms is similar to the previous case, so the bound is yet another

$$\leqslant \text{const} \cdot |h|_0 \, M^{\varepsilon j} \tag{3.59}$$

which proves (a), and (c) for $s = 0$. For $s = 1$, we need not apply Lemma 2.42: the derivative w.r.t. the external momentum can act only on propagators associated to lines of W, and effectively produces a factor M^{-j^*} (it cannot act on h, since h is in a string in the present case). The D_h acting on the string causes a factor M^{-j} as compared to the ordinary scaling behavior, which we take outside. The bound now follows by the argument between (3.49) and (3.53).

Some of the bookkeeping of this $P = 0$ case could have been avoided by normal ordering, but the normal ordering prescription depends on e, and thus D_h would have produced similar terms there. Also, normal ordering does not remove the GST graphs, so $P > 0$ has to be dealt with anyway.

Up to now, the sum over trees was not really used, since the only term where a resummation of the partition of unity was necessary was the lowest order term. Let $P > 0$, and assume that the IH holds for depth $P' \leqslant P - 1$. Now there are also two-legged subdiagrams on which D_h can act. It can also act on the projections ℓ or $1 - \ell$ in front of them since projection on S depends on e. For every $t \sim G$, we construct the graph G' as in Remark 2.45. We rearrange the sum over trees t as follows. Every $t \sim G$ that gives rise to the same G' can be split into the tree t' rooted at ϕ and the subtrees t_w of t associated to every vertex w of G'. Thus, at fixed G', the sum over all $t \sim G$ splits into one over all $t' \sim G'$ and given $t' \sim G'$, there is a sum over trees t_w rooted at w for every vertex w of G'. Blocking the sum in that way, we have for every G' and t' the vertex functions

$$\Phi_w = \sum_{t_w \sim G_w} \prod_{f \in t_w} \frac{1}{n_f!} F_w \tag{3.60}$$

with F_w given by (2.125). For two-legged vertices w, Φ_w has the same structure as the left side of (3.41), but the depth of (t_w, G_w) is smaller than P

for all $t_{w'}$ contributing to the sum for $\Phi_{w'}$. This is the basis of the induction scheme.

Case 4. $P>0$, $E(G)=2, 4$, and $\tilde{G}(\phi)$ overlapping. Then $(G')^{\sim}(\phi)$ is overlapping as well. For the derivative of a propagator we again use

$$|D_h C_{j_l}(p)| \leqslant \text{const} \cdot 1_{j_l}(p)\, M^{-2j_l}\, |h|_0 \tag{3.61}$$

If the derivative acts on an r-fork, we write $\Phi_{w'} = (1-\ell)\, T$ and use

$$D_h(1-\ell)\, T = (1-\ell)\, D_h T + \ell\left(\frac{h}{\mathscr{D}_u e}\, \mathscr{D}_u T\right) \tag{3.62}$$

to isolate the term where the renormalization cancellation gets lost. In the second term, we associate the factor

$$\zeta = \ell\left(\frac{h}{\mathscr{D}_u e}\, \mathscr{D}_u T\right) \tag{3.63}$$

to one of the lines going into $G_{w'}$. By Theorem 2.46, applied to T,

$$|\zeta|_0 \leqslant \frac{|h|_0}{u_0}\, |u|_0\, |T|_1 \leqslant \text{const} \cdot |h|_0 \tag{3.64}$$

By the IH (b) or (c),

$$\begin{aligned} |1_{j_{\pi(w')}}(1-\ell)\, D_h T|_0 &\leqslant \text{const} \cdot M^{j_{\pi(w')}}\, |D_h T|_1 \\ &\leqslant |h|_0\, M^{j_{\pi(w')}} \sum_{j' \geqslant j_{\pi(w')}} \text{pol}(j')\, M^{(\varepsilon-1)j'} \\ &\leqslant |h|_0\, \text{pol}(j_{\pi(w')})\, M^{\varepsilon j_{\pi(w')}} \end{aligned} \tag{3.65}$$

Adding (3.64) and (3.65), we obtain

$$|D_h \Phi_{w'}|_0 \leqslant \text{const} \cdot |h|_0 \tag{3.66}$$

so, compared to the usual behavior (2.155), we have lost a factor $M^{j_{\pi(w')}}$, which is the same as saying that there is an extra factor M^{-j_l} for one of the two external legs of the graph $G_{w'}$, just as if the D_h had acted on that leg.

Similarly, if the derivative acts on a c-fork, $\Phi_{w'} = \ell T$,

$$D_h \ell T = \ell D_h T - \ell\left(\frac{h}{\mathscr{D}_u e}\, \mathscr{D}_u T\right) \tag{3.67}$$

where

$$T = \sum_{I \leqslant j' \leqslant j_{\pi(w)}} \sum_{t_w \sim G_w} \prod_{f \in t_w} \frac{1}{n_f!} \sum_{J \in \mathscr{J}(t_w, j')} Val(G_w^J) \tag{3.68}$$

application of the IH (a) to the first term and Theorem 2.46 to the second term yields $|D_h \ell T|_0 \leqslant \text{const} \cdot |h|_0$. Compared to (2.162), the derivative has again cost us a factor $M^{j_{\pi(w)}}$. Again, we associate a factor $z_l = M^{-j_l}$ to one of the external legs of G_w.

For an SSI, there is the same factor. For a four-legged vertex, there is a factor $\text{pol}(j_{\pi(w)}) \, M^{-j_{\pi(w)}}$ by the IH (b) and (c).

To summarize, the effect of D_h on the scale behavior, as compared to the power counting behavior (2.162) and (2.163), is accounted for by an additional factor

$$z_l = M^{-j_l} \tag{3.69}$$

on a line of G'. By construction of G', $M^{-j_l} \leqslant M^{-j}$, so, by Theorem 2.40 [similar to (2.166)]

$$|Val(G'^J)|_0 \leqslant \text{const} \cdot M^{j(D_\phi(G') + \varepsilon - 1)} \prod_{\substack{f \in t' \\ f > \phi}} M^{D_f(G')(j_f - j_{\pi(f)})} \prod_{w \in V_2(G')} M^{j_{\pi(w)}} \tag{3.70}$$

The proof that the scale sum over $J \in \mathscr{J}(t, j)$ converges is given following (2.166), so (b) holds for $s = 0$. For $E = 2$, the sum over j converges because of the remaining $M^{\varepsilon j}$, which proves (3.33) and (a) and (by the usual dominated convergence argument) also (3.34).

For $s = 1$, we apply an extra derivative with respect to the external momentum q before taking $|\cdot|_0$. All we have to show is that its effect can be bounded by a factor $\text{const} \cdot M^{-j}$. For its action on a propagator, this follows from Lemma 2.3(iii), and for its action on a vertex function Φ_w that is not affected by D_h, it follows from Theorem 2.46. For its action on D_h of a vertex function coming from an r-fork or a four-legged subdiagram of G, it follows from the IH (b) and (c), since the scales of these vertices are summed above $j_{\pi(w)}$. However, we have to avoid two derivatives on any c-fork, and also prevent the derivative from acting on h. The only case when two derivatives can act on a c-fork is when D_h acts on the c-fork and $\partial/\partial q_i$ acts on the same c-fork. The latter derivative can be removed by an integration by parts because G is 1PI (and then cannot act on h), so the bound for $s = 1$ follows as in the proof of Theorem 2.46.

Case 5. $P > 0$, $E(G) = 4$, and $\tilde{G}(\phi)$ nonoverlapping. The bound (c) is proven as above, omitting the parts used to get $\varepsilon > 0$.

Case 6. $P > 0$, $E(G) = 2$, and $\tilde{G}(\phi)$ nonoverlapping. Since G' is a quotient graph of G that contains all lines $l \in L(G)$ with $j_l = j$, $\widetilde{G'}(\phi) = \tilde{G}(\phi)$ is a GST graph, and so is $\widetilde{G'}(\tau'_\phi)$, where τ'_ϕ is the maximal nonoverlapping subtree of t' rooted at ϕ [Lemma 2.31(ii)]. The value of $\widetilde{G'}(\tau'_\phi)$ is an integral of the form given in Remark 2.41 (from which we now take the notation). By construction of τ'_ϕ, the vertex function $\mathscr{U}_{v_1}$ belongs to a subgraph H of G', of scale j^* such that G' overlaps on scale j^*. D_h can act on $\mathscr{U}_{v_1}$ or on one of the strings S_i. In the former case, the bound follows by (3.51)–(3.53), because (3.69) applies due to the volume gain at the scale j^* where the derivative acts, and because strings of two-legged subdiagrams satisfy $|S_i(p)| \leqslant \text{pol}(j)\, M^{-j} 1_j(p)$ by (2.162).

Let D_h act on one of the strings. We call $j^{(i)}$, defined in Remark 2.41, the scale of the string S_i. We do the case $s = 1$ first. We choose the spanning tree as in Lemma 2.38; then the additional derivative with respect to the external momentum q acts only on $\mathscr{U}_{v_1}$, i.e., at a scale where there is an improvement factor $M^{\varepsilon j^*}$, and it cannot act on h. The effect of D_h is again accounted for by a factor $M^{-j^{(i)}} \leqslant M^{-j}$ on one of the lines. Taking the M^{-j} in front, (c), for $s = 1$, follows by (3.51)–(3.53).

For the final case $s = 0$, we consider two situations, sketched in Fig. 20, separately.

(A) There is a string of scale j (i.e., root scale) on which D_h does not act. (This is the case if there are at least two strings of scale j or if D_h acts on a string S' of higher scale.)

(B) There is only one string on root scale, and D_h acts on it.

(A) By (3.69) (and since all insertions in a string are two-legged), there is effectively an $M^{-j'}$ and a factor $h(p')$, where p' is the loop momentum

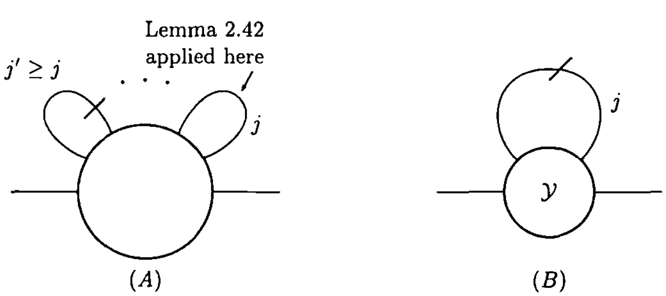

Fig. 20. The case $\tilde{G}(\phi)$ nonoverlapping and D_h acting at root scale.

of S'. We apply the Taylor expansion procedure of Lemma 2.42 to the string of scale j on which D_h did not act (see Fig. 20A). Although this produces derivatives on other factors in the expression for $Val(G)$, there are no ∇h terms, because p' is an independent loop momentum. The Taylor expansion generates two kinds of terms, one from acting on the r-forks, bounded by

$$\text{const} \cdot M^{2j} \, |T|_2 \, |\mathcal{U}_{v_1}|_0 \leqslant \text{pol}(j) \, M^{2j} M^{(\varepsilon-1)j} \, |\mathcal{U}_{v_1}|_0 \tag{3.71}$$

by Theorem 2.46 (this includes the sum over the scale of T as well as the loop integral) and thus convergent, and one where $\mathcal{U}_{v_1}$ gets differentiated,

$$1_j(p) \left| \frac{\partial}{\partial p} \mathcal{U}_{v_1}(q, p, ...) \right| \tag{3.72}$$

The scale balance is identical to the $P=0$ case, so (3.57)–(3.59) hold, with const replaced by pol(j). This proves (c) and (a).

(B) In this case, the vertex v_1 is a four-legged vertex of G' with a vertex function $\mathcal{Y} = \mathcal{U}_{v_1}$, to which the IH applies. We thus have to bound $\sum_{j \geqslant I} X_j(q)$, where

$$X_j(q) = \int d^{d+1}p \, \mathcal{Y}(q, p) \, D_h S(p) \tag{3.73}$$

In some of the following cases, we need to resolve the four-legged graph to which the $\mathcal{Y}$ is associated (unless it is a vertex of scale zero, which behaves as a vertex with improvement factor M^j whenever a derivative acts on it). The vertex function $\mathcal{Y}$ is given by the scale sum

$$\mathcal{Y}(q, p) = \sum_{i \geqslant j+1} \mathcal{Y}_i(q, p) \tag{3.74}$$

with $|\mathcal{Y}_i|_s \leqslant \text{pol}(i) \, M^{-si}$ (see Fig. 20). Note that by Theorem 2.40 and by construction of τ'_ϕ, there is a volume gain $M^{\varepsilon j^*}$ in the entire integral for X_j. S is a string of length n,

$$S(p) = \cdot \sum_{\substack{j_1, ..., j_{n-1} = j \\ \min\{j_1, ..., j_{n-1}\} = j}}^{j+1} \left(\prod_{k=1}^{n-1} C_{j_k}(p_0, e(\mathbf{p})) \, \mathcal{P}_k T_k(p) \right) C_{j_n}(p_0, e(\mathbf{p})) \tag{3.75}$$

The T_k are scale sums of 1PI two-legged insertions and thus dependent on j and $\mathcal{P}_k \in \{1, \ell, 1-\ell\}$. The T_k obey the bounds of Theorem 2.46 and, as graphs of depth $P' \leqslant P-1$, also the IH. The undifferentiated string obeys $|S(p)| \leqslant 1_j(p) \, \text{pol}(j) \, M^{-j}$ by (2.162).

The form of the scale sum in (3.75) is due to the sum over all trees $t' \sim G'$, which contains a sum over all these assignments. This is the point where the tree sum is necessary, as will be seen when the partition of unity is resummed.

If D_h acts on an insertion from an r-fork, we get the two terms of (3.62). The first one is bounded using (3.65), so in this term the D_h changes the root scale behavior from M^j to $M^{\varepsilon j}$. For the second term, call $U = (h/\mathscr{D}_u e)\, \mathscr{D}_u T$. By (3.64), $|U|_0 \leqslant \text{const} \cdot |h|_0$, including the sum over the scale of T. This amounts to a loss of M^{-j}. Apply Lemma 2.42 to the string S to extract another M^{j-j^*} and use the volume gain of $M^{j^*\varepsilon}$. Note that the Taylor expansion does not produce any derivatives of h since (in the notation of Lemma 2.42) $\ell U(p) = \tilde{U}(0, 0, \omega)$ does not depend on r and φ. The loss of the renormalization cancellation would make the scale sum over j marginal, but the extra $M^{\varepsilon j}$ makes it convergent. This proves (c) and (a) for this contribution.

The case of D_h acting on a c-fork is similar since

$$D_h \ell T = \ell(D_h T) - \ell\left(\frac{h}{\mathscr{D}_u e} \mathscr{D}_u T\right)$$

The second term is bounded by $|h|_0 \operatorname{pol}(j)\, M^{\varepsilon j}$ by (2.153). The IH applies to the first term, but we have to make up for the loss of the usual factor $M^{(1-s)j}$ of (2.162). Therefore, after applying D_h, we use Lemma 2.42 in the same way as for the ℓU term of the just treated r-fork case. The Taylor expansion is such that the term $\ell D_h T$ is treated like a constant. Thus, after Taylor expanding and collecting the gain $M^{\varepsilon j}$, we find that $|\ell D_h T|_0$ appears in the bound. The IH applies and implies (c) and (a) for this contribution. For SSI, write $1 = 1 - \ell + \ell$ and treat the two terms as above.

Finally, D_h can act on a propagator of scale j (or $j+1$). Now we may assume that there are no c-forks or same scale insertions or r-forks of scale below, for example, $j+7$ on the string S, since otherwise (c) and (a) follow immediately from the improved power counting behavior (2.152), which suffices by itself to control the M^{-j} from the action of D_h. The strategy is now similar to that of the lowest order example. To get (3.33), we apply (3.7). There are three terms: the term containing $\nabla \cdot (hu/\mathscr{D}_u e)$ has the same scale behavior as if the derivative had not acted (but contains a ∇h). The second term is when $u \cdot \nabla = \mathscr{D}_u$ acts on an r-fork $(1-\ell)\, T$. Since $\mathscr{D}_u \ell = 0$, the result is $(1-\ell)\, \mathscr{D}_u T + \ell \mathscr{D}_u T$. By Theorem 2.46, the first summand has a net $M^j \sum_{j' > j} M^{j'(1+\varepsilon-2)} = M^{\varepsilon j}$, and the second is treated by Lemma 2.42, as above. The third term is when $\mathscr{D}_u$ acts on $\mathscr{Y}$. In that case, we resolve the corresponding subgraph and proceed as in the case $\mathscr{U}_{v_1}$. So in all cases, the scale behavior deteriorates from M^j only to $M^{\varepsilon j}$, which proves (3.33), and, by the same argument as in the proof of Theorem 2.46, also (3.34). For the proof of (a) and (c) we must again consider two terms, because

$$D_h C_j(p) = h(\mathbf{p})(i\partial_0 C_j(p) - 2M^{-2j} f_j'(p))$$

In the string S, actually a product $C_j(p)^m C_{j+1}(p)^{n-m}$ appears, where $m \geqslant 1$ depends on the scale assignments, and the relevant formula is

$$\begin{aligned} D_h\left(\prod_{l=1}^{n} C_{j_l}(p)\right) &= h(\mathbf{p})\, i\partial_0 \prod_{l=1}^{n} C_{j_l}(p) \\ &\quad - \frac{2h(\mathbf{p})}{(ip_0 - e(\mathbf{p}))^{n-1}} \left[\frac{\partial}{\partial x}\left(\prod_{l=1}^{n} f(M^{-2j_l}x)\right)\right]_{x = p_0^2 + e(\mathbf{p})^2} \end{aligned} \tag{3.76}$$

In the contribution of the first term to (3.76), we integrate by parts in p_0. There is no boundary term. h depends only on $\mathbf{p}$, so ∂_0 can act only on the vertex function $\mathcal{Y}$ or on an r-fork. In the former case, we combine the known behavior $|\mathcal{Y}|_1 \leqslant \text{pol}(j)\, M^{-j^*}$, Theorem 2.40, to get the volume factor $M^{\varepsilon j^*}$, and Theorem 2.46 for the T_k and the standard bounds for C_j,

$$\begin{aligned} \left| \int d^{d+1}p\, S(p) \frac{\partial}{\partial p_0} \mathcal{Y}(q,p)\, h(\mathbf{p}) \right| &\leqslant \text{pol}(j)\, |h|_0 \cdot M^{j(\varepsilon-1)} M^j \\ &\leqslant |h|_0\, \text{pol}(j)\, M^{j\varepsilon} \end{aligned} \tag{3.77}$$

to see that the scale sum converges [more precisely, we apply the procedure of (3.49)–(3.53), which by now should be routine].

If ∂_0 acts on an r-fork, we get two terms from $\partial_0(1-\ell)\,T = \partial_0 T = (1-\ell)\,\partial_0 T + \ell \partial_0 T$. Recalling that T is given by a scale sum $T = \sum_{i>j} T_i$, recalling (2.97), and using Lemma 2.7, we have

$$|(1-\ell)\,\partial_0 T|_{0,j} \leqslant \sum_{i>j} |(1-\ell)\,\partial_0 T_i|_{0,j} \leqslant \frac{\sqrt{2}}{u_0} M^j \sum_{i>j} |T_i|_2 \tag{3.78}$$

By Theorem 2.46(i), this is

$$\begin{aligned} &\leqslant \frac{\sqrt{2}}{u_0} M^j M^{\varepsilon/2} Q(T)\, |v|_2^{|V(T)|} \sum_{i>j} \lambda_{nT}\left(i, \frac{\varepsilon}{2}\right) M^{i(\varepsilon-1)} \\ &\leqslant \frac{\sqrt{2}}{u_0} M^j M^{\varepsilon/2} Q(T)\, |v|_2^{|V(T)|} M^{j(\varepsilon-1)} \lambda_{nT}\left(j, \frac{\varepsilon}{2}\right) \frac{1}{1-M^{\varepsilon-1}} \\ &\leqslant \text{const} \cdot M^{j\varepsilon} \end{aligned} \tag{3.79}$$

The second term is not as easy because there is no reason for $\ell\partial_0 T$ to be small. Here we have to use another integration by parts, and in some terms an additional resummation, as follows. We have to bound

$$X=\int dp\, h(\mathbf{p})\, \mathcal{Y}(q,p)(\ell\partial_0 T^{(1)})(p) \prod_{l=1}^{n} C_{j_l}(p_0, e(\mathbf{p})) \prod_{r=2}^{n-1} (1-\ell)\, T^{(r)}(p) \quad (3.80)$$

for $n\geqslant 2$ (otherwise, such a term does not occur). Superficially, the scale sum of this looks divergent, but we can use that $\ell\partial_0 T$ is independent of p_0 to do another integration by parts. We rewrite X as

$$X=\int dp \frac{h(\mathbf{p})\,\mathcal{Y}(q,p)}{(ip_0-e(\mathbf{p}))^n} (\ell\partial_0 T^{(1)})(p) \prod_{l=1}^{n} f_{j_l}(p) \prod_{r=2}^{n-1} (1-\ell)\, T^{(r)}(p) \quad (3.81)$$

and use that for $n\geqslant 2$

$$\frac{1}{(ip_0-e(\mathbf{p}))^n}=\frac{i}{n-1}\frac{\partial}{\partial p_0}\frac{1}{(ip_0-e(\mathbf{p}))^{n-1}} \quad (3.82)$$

to get

$$X=\frac{-i}{n-1}\int \frac{dp}{(ip_0-e(\mathbf{p}))^{n-1}}\frac{\partial}{\partial p_0} \times\left(h(\mathbf{p})\,\mathcal{Y}(q,p)(\ell\partial_0 T^{(1)})(p) \prod_{l=1}^{n} f_{j_l}(p) \prod_{r=2}^{n-1} (1-\ell)\, T^{(r)}\right) \quad (3.83)$$

h and $\ell\partial_0 T^{(1)}$ do not depend on p_0, so

$$X=\frac{-i}{n-1}\int dp \frac{h(\mathbf{p})(\ell\partial_0 T^{(1)})(p)}{(ip_0-e(\mathbf{p}))^{n-1}} \times\frac{\partial}{\partial p_0}\left(\mathcal{Y}(q,p) \prod_{l=1}^{n} f_{j_l}(p_l) \prod_{r=2}^{n-1} (1-\ell)\, T^{(r)}(p)\right) \quad (3.84)$$

If the derivative were not there, the integral would obey the ordinary power counting bound. We analyze the effect of ∂_0. If the derivative acts on $\mathcal{Y}$, we proceed as above to see that there remains a factor $M^{j\varepsilon}$. We postpone the treatment of the term containing $\partial_0 \prod f_{j+l}(p)$ to the next (and final) case. The derivative acting on $\prod(1-\ell)\, T^{(r)}$ produces a sum of terms similar to the one we started with, but with number of r-forks on the string decreased by one. Thus we may proceed iteratively to remove all these terms, so that it remains to estimate (having renumbered the T's)

$$\int dp \frac{\mathcal{Y}(q,p)\, h(\mathbf{p}) \prod_{r=1}^{k} (\ell \partial_0 T^{(r)})(p)}{(ip_0 - e(\mathbf{p}))^{n-k}} \prod_{r=k+1}^{n-1} (1-\ell)\, T^{(r)}(p) \frac{\partial}{\partial p_0} \prod_{l=1}^{n} f_{j_l}(p_l) \tag{3.85}$$

for $k \leqslant n-1$. This will be done by resummation, and the procedure is similar for all k, and similar to the procedure to deal with the second term in (3.76), which we discuss now.

We have to bound the integral

$$\begin{aligned} Y(q) = -2 \sum_{0 > j \geqslant I} \int d^{d+1}p\, \mathcal{Y}(q,p) \frac{h(\mathbf{p})}{(ip_0 - e(\mathbf{p}))^{n-1}} \\ \times \sum_{\substack{j_1, \ldots, j_n = j \\ \min\{j_1, \ldots, j_n\} = j}}^{j+1} \left(\prod_{k=1}^{n-1} (1-\ell)\, T_k(p) \right) \\ \times \sum_{l=1}^{n} M^{-2j_l} f'(M^{-2j_l}(p_0^2 + e(\mathbf{p})^2)) \prod_{r \neq l} f(M^{-2j_r}(p_0^2 + e(\mathbf{p})^2)) \end{aligned} \tag{3.86}$$

by resumming the partition of unity on line number l. To this end, we first rearrange the sum over the j_k by using that for all $k \in \{1, \ldots, n\}$

$$\begin{aligned} \bigcup_{j=I}^{-1} \{(j_1, \ldots, j_n) \colon j_k \in \{j, j+1\} \cap \{I, \ldots, -1\}, \min\{j_1, \ldots, j_n\} = j\} \\ = Z_k(I) \cup \bigcup_{j=I+1}^{-1} M_k(j) \end{aligned} \tag{3.87}$$

where

$$\begin{aligned} M_k(j) = \{(j_1, \ldots, j_n) \colon j_k = j, j_r \in \{I, \ldots, -1\} \\ \text{and for all } r, s \in \{1, \ldots, n\}, |j_r - j_s| \leqslant 1\} \end{aligned} \tag{3.88}$$

$$Z_k(j) = \{(j_1, \ldots, j_n) \colon j_k = j \text{ and } j_r \in \{j, j+1\} \text{ for } r \neq k\}$$

To see this, we note that the left side is $Z = Z_1 \cup Z_2$ with, for each fixed k,

$$\begin{aligned} Z_1 = \bigcup_{j=I}^{-1} \{(j_1, \ldots, j_n) \colon j_k = j, j_r \in \{j, j+1\} \text{ for } r \neq k\} \\ = Z_k(I) \cup \bigcup_{j=I+1}^{-1} Z_k(j) \end{aligned} \tag{3.89}$$

and

$$Z_2 = \bigcup_{j=I}^{-2} \{(j_1,..., j_n): j_k = j+1, j_r \in \{j, j+1\} \text{ for } r \neq k, \min\{j_1,..., j_n\} = j\}$$

$$= \bigcup_{i=I+1}^{-1} \{(j_1,..., j_n): j_k = i, j_r \in \{i-1, i\} \text{ for } r \neq k, \min\{j_1,..., j_n\} = i-1\}$$

$$= \bigcup_{j=I+1}^{-1} V_k(j) \tag{3.90}$$

Here

$$V_k(j) = \{(j_1,..., j_n): j_k = j, j_r \in \{j-1, j\} \text{ for } r \neq k \text{ and } \min\{j_1,..., j_n\} = j-1\}$$

We have

$$Z = Z_k(I) \cup \bigcup_{j=I+1}^{-1} (Z_k(j) \cup V_k(j)) \tag{3.91}$$

Finally,

$$Z_k(j) \cup V_k(j) = \{(j_1,..., j_n): j_k = j \text{ and either } j_r \in \{j, j+1\}\ \forall r \neq k \text{ or } j_r \in \{j-1, j\}\ \forall r \neq k \text{ with } \min\{j_1,..., j_n\} = j-1\} \tag{3.92}$$

The condition $\min\{j_1,..., j_n\} = j-1$ implies that the sequence $(j,..., j)$ appears only once [i.e., $Z_k(j) \cap V_k(j) = \emptyset$]. The result is then obviously equivalent to $|j_r - j_s| \leqslant 1$ for all r, s.

We apply this to Y as follows. Note that all T_k and $\mathscr{Y}$ depend on j (unless $\mathscr{Y}$ is just a vertex of scale zero). For all $r \in \{1,..., n-1\}$, T_r actually depends on the scale j_r of a line going into T_r. Fix all j_r for $r \neq k$; then the scales of all T_s are fixed. If $\mathscr{Y}$ depends on j, it is of the form

$$\mathscr{Y}(q, p) = \sum_{i>j} \mathscr{Y}_i(q, p) \tag{3.93}$$

We interchange the sums over i and j,

$$Y(q) = -2 \sum_{0>i\geqslant I} \sum_{i>j\geqslant I} \int d^{d+1}p\, \mathscr{Y}_i(q,p) \frac{h(\mathbf{p})}{(ip_0 - e(\mathbf{p}))^{n-1}}$$
$$\times \sum_{\substack{j_n,\dots,j_n=j \\ \min\{j_1,\dots,j_n\}=j}}^{j+1} \left(\prod_{s=1}^{n-1} (1-\ell)\, T_s(p)\right)$$
$$\times \sum_{l=1}^{n} M^{-2j_l} f'(M^{-2j_l}(p_0^2 + e(\mathbf{p})^2)) \prod_{r\neq l} f(M^{-2j_r}(p_0^2 + e(\mathbf{p})^2)) \tag{3.94}$$

We have

$$\bigcup_{j=I}^{i} \{(j_1,\dots,j_n): j_k \in \{j, j+1\} \cap \{I,\dots,-1\}, \min\{j_1,\dots,j_n\} = j\}$$
$$= V_k(i+1) \cup Z_k(I) \cup \bigcup_{j=I+1}^{i} M_k(j) \tag{3.95}$$

as before. Choose $k = l+1$ or $k = l-1$; then

$$\sum_{(j_1,\dots,j_n)\in M_k(j)} \prod_{s=1}^{n-1} (1-\ell)\, T_s(p) \prod_{r\neq l} f(M^{-2j_r}(p_0^2 + e(\mathbf{p})^2))$$
$$\times M^{-2j_l} f'(M^{-2j_l}(p_0^2 + e(\mathbf{p})^2))$$
$$= \sum_{\substack{(j_1,\dots,j_n)\in M_k(j) \\ j_l \text{ not summed}}} \left(\prod_{s=1}^{n-1} (1-\ell)\, T_s(p)\right) \sum_{j_l=j-1}^{j+1} \left[\frac{\partial}{\partial x} f(M^{-2j_l}x)\right]_{x=p_0^2+e(\mathbf{p})^2}$$
$$\times \prod_{r\notin\{k,l\}} f(M^{-2j_r}(p_0^2 + e(\mathbf{p})^2))\, f(M^{-2j}(p_0^2 + e(\mathbf{p})^2)) \tag{3.96}$$

By momentum conservation and the support properties of the f, (2.16), we can now extend the sum over j_l to the entire interval $\{I,\dots,i\}$. Using

$$\sum_{j=I}^{i} f(M^{-2j}x) = a(M^{-2I}x) - a(M^{-2(i+1)}x) \tag{3.97}$$

[see (2.12) and following] and calling $E^2 = p_0^2 + e(\mathbf{p})^2$, we have

$$f(M^{-2j}E^2) \sum_{j_l=I}^{i} M^{-2j_l} f'(M^{-2j_l}E^2)$$
$$= f(M^{-2j}E^2)(M^{-2I}a'(M^{-2I}E^2) - M^{-2(i+1)}a'(M^{-2(i+1)}E^2)) \tag{3.98}$$

Since $a'(x) \neq 0$ only for $x \in (M^{-4}, M^{-2})$ and $f(x) \neq 0$ only for $x \in (M^{-4}, 1)$,

$$f(M^{-2k}x)\, a'(M^{-2l}x) = 0 \text{ for all } x \text{ unless } k \in \{l, l-1\}$$

Since $j \in \{I, ..., i\}$, the only nonzero terms in the sum over j are $j = I$ and $j = i$. Thus we get four boundary terms in the estimate for Y, two at both the lower and upper ends of the summation region; the two from $B_{1,k}(i)$ and $B_{2,k}(I)$ are similar to the following two, which we discuss in detail.

If $j = I$, we estimate

$$\begin{aligned}
&\Bigg| 2 \sum_{0 > i \geq I} M^{-2I} \int d^{d+1}p\, \mathscr{Y}_i(q,p) \frac{h(\mathbf{p})}{(ip_0 - e(\mathbf{p}))^{n-1}} \\
&\quad \times \sum_{\substack{(j_1, ..., j_n) \in M_k(I) \\ j_l \text{ not summed}}} \prod_{s=1}^{n-1} (1 - \ell)\, T_s(p) \\
&\quad \times \prod_{r \notin \{k,l\}} f(M^{-2j_r}(p_0^2 + e(\mathbf{p})^2)) \Bigg| \\
&\quad \leq \text{const} \cdot |h|_0 \sum_{i=I}^{-1} \int d^{d+1}p\, |\mathscr{Y}_i(q,p)| \frac{1}{|ip_0 - e(\mathbf{p})|^{n-1}} \prod_{s=1}^{n-1} |(1-\ell)\, T_s(p)| \\
&\quad \times 1(|ip_0 - e(\mathbf{p})| \in [M^{I-2}, M^I])
\end{aligned} \tag{3.99}$$

We now use Taylor expansion for all the r-forks as in the proof of Theorem 2.46, to bound this by

$$\begin{aligned}
&\leq \text{const} \cdot |h|_0 \left(\prod_{s=1}^{n-1} M^I\, |T_s|_1 \right) M^{-2I} M^{-(n-1)(I-2)} \sum_{i=I}^{-1} \int d^{d+1}p\, |\mathscr{Y}_i(q,p)| \\
&\quad \times 1(|ip_0 - e(\mathbf{p})| \in [M^{I-2}, M^I]) \\
&\leq \text{const} \cdot |h|_0\, R \prod_{s=1}^{n-1} |T_s|_1
\end{aligned} \tag{3.100}$$

where

$$R = M^{-2I} \sup_q \sum_{i=I}^{-1} \int d^{d+1}p\, |\mathscr{Y}_i(q,p)|\; 1(|ip_0 - e(\mathbf{p})| \in [M^{I-2}, M^I]) \tag{3.101}$$

By maximality of τ'_ϕ, we know that there is a volume improvement on scale i, so

$$R \leq M^{-2I} \cdot \text{const} \cdot \sum_{i=I}^{-1} M^{\varepsilon i} M^{2I} \leq \text{const} \tag{3.102}$$

It remains to be shown that the product over $|T_s|_1$ stays bounded, i.e., that there are no factors I. This can be seen by estimating the last sum in (2.155) differently: by Lemma 2.44(v),

$$\begin{aligned}\sum_{j_w > j_{\pi(w)}} \lambda_n(j_w, \varepsilon/2)\, M^{j_w \varepsilon} &= \sum_{l=1}^{|j_{\pi(w)}|-1} \lambda_n(-l, \varepsilon/2)\, M^{-\varepsilon l} \\ &\leqslant \sum_{l=1}^{\infty} (a_n l^n + b_n)\, M^{-\varepsilon l} \\ &\leqslant \text{const} \cdot (n, \varepsilon) \end{aligned} \tag{3.103}$$

independently of $j_{\pi(w)}$, so, inserting this into (2.155), we get $|T_s|_1 \leqslant \text{const}$ for all s. Thus this term is bounded.

At the upper summation end $j = i$, we have to estimate

$$\begin{aligned}&\Bigg| \sum_{0 > i \geqslant I} \int d^{d+1}p\, \mathscr{Y}_i(q, p) \frac{h(\mathbf{p})}{(ip_0 - e(\mathbf{p}))^{n-1}} \\ &\quad \times M^{-2i} \sum_{\substack{(j_1, \ldots, j_n) \in M_k(i) \\ j_l \text{ not summed}}} \prod_{s=1}^{n-1} (1 - \ell)\, T_s(p) \prod_{r \notin \{k, l\}} f(M^{-2j_r}(p_0^2 + e(\mathbf{p})^2)) \Bigg| \\ &\leqslant \text{const} \cdot |h|_0 \sum_{i=I}^{-1} M^{-2i} \int d^{d+1}p\, |\mathscr{Y}_i(q, p)|\, 1(|ip_0 - e(\mathbf{p})| \in [M^{i-2}, M^i]) \\ &\quad \times M^{-(n-1)(i-2)} \prod_{s=1}^{n-1} (1 - \ell)\, T_s(p) \\ &\leqslant \text{const} \cdot |h|_0 \sum_{i=I}^{-1} M^{-2i} \prod_{s=1}^{n-1} |T_s|_1 \\ &\quad \times \int d^{d+1}p\, |\mathscr{Y}_i(q, p)|\, 1(|ip_0 - e(\mathbf{p})| \in [M^{i-2}, M^i]) \end{aligned} \tag{3.104}$$

By (3.51)–(3.53) without the D_h, the last integral is bounded by $\text{const} \cdot M^{(2+\varepsilon)i}$, so this is

$$\begin{aligned} &\leqslant |h|_0 \sum_{i=I}^{-1} M^{\varepsilon i} \prod_{s=1}^{n-1} |T_s|_1 \\ &\leqslant \text{const} \cdot |h|_0 \quad \blacksquare \end{aligned} \tag{3.105}$$

Remark 3.7. Actually, the following stronger convergence statement holds. Let (t, G) be fixed, $E(G)=2$, G 1PI. Let $|h|_1<\infty$ and for $I'>I$

$$V_I^{I'}(h, t, G)=\sum_{j=I}^{I'-1} \sum_{J \in \mathscr{J}(t, j)} D_h \, Val(G^J) \tag{3.106}$$

Then, as $I \to -\infty$, $V_I^{I'} \to V^{I'}$ with $|V^{I'}|_0 \leqslant |h|_1 \, W^{I'}$ and $W^{I'}$ vanishes as $I' \to -\infty$. In particular, for all $k \geqslant 0$,

$$|V_I(h, t, G)-V_{I-k}(h, t, G)|_0 \leqslant |h|_1 \, \tilde{W}_I \tag{3.107}$$

with $\tilde{W}_I \to 0$ as $I \to -\infty$.

3.3. Convergence of the Derivative

In this section, we prove Theorems 1.6–1.8. Given all the detailed information about the two-legged and four-legged vertices that we have gathered in the last two sections, these proofs will be applications of elementary convergence theorems for absolutely convergent series. For the convenience of the reader, we summarize the results derived so far. Recall the explicit expression (2.76),

$$K_r^I(\mathbf{p})=-\sum_G \sum_{j=I}^{-1} \sum_{t \sim G} \prod_{f \in t} \frac{1}{n_f!} \sum_{J \in \mathscr{J}(t, j)} Val(G^J)(0, \mathbf{P}(\mathbf{p}))$$

For all $r \geqslant 1$, K_r^I converges as $I \to \infty$ in $|\cdot|_1$ to a function $K_r \in C^1(\mathscr{B}, \mathbb{R})$ (see Section 2.7). Let $\mathscr{A}$ be as in Definition (1.53), and $\mathscr{L}$ as defined thereafter. Let e be in $\mathscr{A}$. Since for $I>-\infty$, K_r^I is differentiable in the sense of Fréchet,[8] the derivative $(K_r^I)' \in \mathscr{L}$ can be evaluated as the directional derivative

$$(K_r^I)'(h)=D_h K_r^I(e) \tag{3.108}$$

Fix $r \geqslant 1$, let $I=-n$, and denote $K_r^{-n}=\kappa_n$. Then by (3.33) (summed from $-n-m$ to $-n$), $(\kappa_n)_{n \geqslant 1}$ satisfies

$$\lim_{n \to \infty} \sup_{m>0} \sup_{|h|_1=1} \frac{1}{|h|_1} |\kappa_n'(h)-\kappa_{n+m}'(h)|_0=0 \tag{3.109}$$

for all $m \geqslant 1$; thus it converges in operator norm $\|\cdot\|_{\mathscr{L}}$ to an operator $\kappa_e' \in \mathscr{L}$. By (3.108),

$$\kappa_e'(h)=\lim_{n \to \infty} D_h \kappa_n \tag{3.110}$$

so by (3.35)

$$|\kappa'_e(h)|_0 \leqslant \text{const} \cdot |h|_0 \tag{3.111}$$

So far this was all at fixed $e \in \mathscr{A}$. The constants and bounds depend on u_0, ε, $|e|_2$, $|u|_2$, and the size δ of the neighborhood of S. By definition of $\mathscr{A}$, (1.53), every $e \in \mathscr{A}$ has a neighborhood $\mathscr{U}$ on which these constants are uniform, and thus all the above bounds hold uniformly in e, and also the convergence is uniform. It remains to be shown that κ' is the derivative of κ and that the map $e \to \kappa'_e$ is continuous on $\mathscr{A}$. We first show by an $\varepsilon/3$-argument that $e \mapsto \kappa'_e$ is continuous, as follows. Write

$$\kappa'_{e_1} - \kappa'_{e_2} = \kappa'_{e_1} - \kappa'_n(e_1) + \kappa'_n(e_1) - \kappa'_n(e_2) + \kappa'_n(e_2) - \kappa'_{e_2} \tag{3.112}$$

Let $\varepsilon > 0$; then there is $n \geqslant 1$ such that $\|\kappa'_{e_i} - \kappa'_n(e_i)\| < \varepsilon/3$ for all $e_i \in \mathscr{U}$. Fix $I = -n$ with that property. At fixed I,

$$\|(K^I_r)'\,(e_1) - (K^I_r)'\,(e_2)\| < C_I\,\|e_1 - e_2\| \tag{3.113}$$

C_I grows with I, but we need only one fixed value of I. So for $\|e_1 - e_2\| < \varepsilon/(3C_I)$, $\|\kappa'_n(e_1) - \kappa'_n(e_2)\| < \varepsilon$. It is now easy to see that κ' is the derivative of $K_r = \kappa$ since

$$\begin{aligned}\kappa_n(e_2) - \kappa_n(e_1) &= \kappa'_n(e_1)(e_2 - e_1)\\ &\quad + \int_0^1 ds\, [\kappa'_n((1-s)\,e_1 + se_2) - \kappa'_n(e_1)](e_2 - e_1)\end{aligned} \tag{3.114}$$

so, taking the limit $n \to \infty$, and calling $h = e_2 - e_1$,

$$\kappa(e_1 + h) - \kappa(e_1) = \kappa'_{e_1}(h) + \int_0^1 ds\, (\kappa'_{e_1 + sh} - \kappa'_{e_1})(h) \tag{3.115}$$

The second term is $o(h)$ by continuity of κ', so κ' is indeed the derivative of κ. This proves Theorem 1.6.

Proof of Theorem 1.7. Let e_1 and e_2 be as stated in Theorem 1.7, $K^{(R)}_\lambda(e) = \sum_{s=1}^R \lambda^s K_s(e)$, and $e_1 + K^{(R)}_\lambda(e_1) = e_2 + K^{(R)}_\lambda(e_2)$. Then

$$e_2 - e_1 + \int_0^1 ds\, \frac{\partial}{\partial s} K^{(R)}_\lambda((1-s)\,e_1 + se_2) = 0 \tag{3.116}$$

that is,

$$(\mathbb{1} + \mathbf{L})(e_2 - e_1) = 0 \tag{3.117}$$

with

$$\mathbf{L}(h) = \int_0^1 ds \sum_{s=1}^{B} \lambda^s K_s'(e_1 + s(e_2 - e_1))(h) \tag{3.118}$$

Since $e_1 + s(e_2 - e_1) \in \mathscr{A}$ for all $s \in [0, 1]$,

$$\|K_s'(e_1 + s(e_2 - e_1))\|_{\mathscr{L}} \leqslant C_s \tag{3.119}$$

so $\|\mathbf{L}\|_{\mathscr{L}} < 1$ and $\mathbb{1} + \mathbf{L}$ is injective. Thus $e_2 - e_1 = 0$. ∎

Theorem 1.8 follows from the observation made earlier that any derivative with respect to the external momentum of a Hartree–Fock-type graph will only act on $\hat{v}$. Since $\hat{v} \in C^k$, so is $D_h H_r^l$, and the statement follows from a standard application of the contraction mapping theorem.

APPENDIX A. VOLUME ESTIMATES

In this Appendix, we prove Proposition 1.1 and uniformity of C_{vol} and ε on the set $\mathscr{A}_2$ defined in (1.53). By definition (1.34), the integral I_2 is symmetric in all three arguments, so we may assume that $\eta_3 \geqslant \max\{\eta_1, \eta_2\}$. By choice of C_{vol} we may also assume that η_3 is so small that $|e(\mathbf{p})| \leqslant 2\eta_3$ implies $\mathbf{p} \in U_\delta(S)$, with δ given in Lemma 2.1, and that $\eta_3 \leqslant G_0\, |e|_2/2$. If S has more than one connected component, we may also assume that η_3 is so small that the neighborhood $|e(\mathbf{p})| \leqslant 2\eta_3$ falls into the same number of connected components as S. Let the coordinates (ρ, ω) be as in Lemma 2.1(iv) and denote $\mathbf{p}$ as a function of these coordinates by $\mathbf{p}(\rho, \omega)$; then

$$\begin{aligned}
I_2(\eta_1, \eta_2, \eta_3) \\
&= \sup_{\mathbf{q} \in \mathscr{B}} \max_{v_1, v_2 \in \{-1, 1\}} \int_{-\eta_1}^{\eta_1} d\rho_1 \int_S d\omega_1\, J(\rho_1, \omega_1) \\
&\quad \times \int_{-\eta_2}^{\eta_2} d\rho_2 \int_S d\omega_2\, J(\rho_2, \omega_2)\, 1(|e(v_1 \mathbf{p}(\rho_1, \omega_1) + v_2 \mathbf{p}(\rho_2, \omega_2) + \mathbf{q})| \leqslant \eta_3) \\
&\leqslant 4 \left(\frac{A_0}{u_0}\right)^2 \eta_1 \eta_2 \sup_{\mathbf{q} \in \mathscr{B}} \max_{v_1, v_2 \in \{-1, 1\}} \sup_{|\rho_1|, |\rho_2| \leqslant \eta_3} \int_S d\omega_1 \\
&\quad \times \int_S d\omega_2\, 1(|e(v_1 \mathbf{p}(\rho_1, \omega_1) + v_2 \mathbf{p}(\rho_2, \omega_2) + \mathbf{q})| \leqslant \eta_3)
\end{aligned} \tag{A.1}$$

By Lemma 2.1 and the mean value theorem

$$|e(v_1 \mathbf{p}(\rho_1, \omega_1) + v_2 \mathbf{p}(\rho_2, \omega_2) + \mathbf{q}) - e(v_1 \mathbf{p}(0, \omega_1) + v_2 \mathbf{p}(0, \omega_2) + \mathbf{q})| \leqslant 2\, |e|_1\, \eta_3/u_0 \tag{A.2}$$

for all ρ_1, ρ_2 with $|\rho_i| \leqslant \eta_3$. Thus

$$|e(v_1 \mathbf{p}(\rho_1, \omega_1) + v_2 \mathbf{p}(\rho_2, \omega_2) + \mathbf{q})| \leqslant \eta_3 \tag{A.3}$$

implies

$$|e(v_1 \mathbf{p}(0, \omega_1) + v_2 \mathbf{p}(0, \omega_2) + \mathbf{q})| \leqslant (1 + 2\, |e|_1/u_0)\, \eta_3 \tag{A.4}$$

hence

$$I_2(\eta_1, \eta_2, \eta_3) \leqslant 4(A_0/u_0)^2\, \eta_1 \eta_2\, W((1 + 2\, |e|_1/u_0)\, \eta_3) \tag{A.5}$$

with

$$W(\eta) = \sup_{\mathbf{q} \in \mathscr{B}} \max_{v_1, v_2 \in \{-1, 1\}} \int_S d\omega_1 \int_S d\omega_2 \times 1(|e(v_1 \mathbf{p}(0, \omega_1) + v_2 \mathbf{p}(0, \omega_2) + \mathbf{q})| \leqslant \eta) \tag{A.6}$$

Thus the function $W(\eta)$ contains the improvement over ordinary power counting. The following lemma implies the bound (1.33) for I_2 with

$$C_{\text{vol}} = Z_3(1 + 2\, |e|_1/u_0)^{\varepsilon} \tag{A.7}$$

where Z_3 is a constant that depends only on Z_0, Z_1, ρ, κ, g_0, and $|e|_2$.

Lemma A.1. $W(\eta) \leqslant Z_3 \eta^{\varepsilon}$.

Proof. Let $\beta \in (0, 1)$ and $\mathscr{T} \subset S \times S$ be the set where the intersection is transversal,

$$\mathscr{T} = \{(\omega_1, \omega_2) \in S \times S\colon [1 - (n(\omega_1) \cdot n(\omega_2))^2]^{1/2} \geqslant \eta^{1-\beta}\} \tag{A.8}$$

and $\mathscr{E}$ its complement, $\mathscr{E} = S \times S \backslash \mathscr{T}$, and split $W(\eta) = T(\eta) + E(\eta)$ into the contributions from these two sets. The idea is that if $(\omega_1, \omega_2) \in \mathscr{T}$, then the tangent spaces $T_{\omega_1} S$ and $T_{\omega_2} S$ span $\mathbb{R}^d$, $T_{\omega_1} S + T_{\omega_2} S = \mathbb{R}^d$, and therefore a combination of ω_1 and ω_2 will be transversal to S, and that $\mathscr{E}$ has small measure because of Assumption A3. β will be chosen at the end.

Fix any $(\bar{\omega}_1, \bar{\omega}_2) \in \mathscr{T}$, and for $i \in \{1, 2\}$, let $T_i = T_{\bar{\omega}_i} S$ as a subspace of $\mathbb{R}^d$ (in other words, $T_i = \{x \in \mathbb{R}^d\colon n(\bar{\omega}_i) \cdot x = 0\}$). By transversality, $T_1 \cap T_2$ is a proper subspace of both T_1 and T_2. Choose an ONB $a_1, \ldots, a_{d-1}$ for T_1

such that a_1 is orthogonal to $T_1 \cap T_2$, and an ONB $b_1,..., b_{d-1}$ for T_2 such that b_1 is orthogonal to $T_1 \cap T_2$; then $|a_1 \cdot b_1| = |n(\bar{\omega}_1) \cdot n(\bar{\omega}_2)|$. Let

$$a_d = \frac{b_1 - (b_1 \cdot a_1)\, a_1}{[1-(n(\omega_1)\cdot n(\omega_2))^2]^{1/2}} \tag{A.9}$$

For $(\alpha_1', \alpha_2,..., \alpha_{d-1})$ in a neighborhood of the origin, let $\omega_1(\alpha_1', \alpha_2,..., \alpha_{d-1}) \in S$ be the image of $\alpha_1' a_1 + \sum_{i=2}^{d-1} \alpha_i a_i \in T_{\bar{\omega}_1} S$ under the exponential map. Similarly let $\omega_1(\beta_1,..., \beta_{d-1}) \in S$ be the image of $\sum_{i=1}^{d-1} \beta_i b_i \in T_{\bar{\omega}_2} S$. The Jacobian $\partial(\omega_1, \omega_2)/\partial(\alpha_1', \alpha_2,..., \beta_{d-1})$ is bounded by a constant in a neighborhood of the origin. Make the further change of variables substituting

$$\alpha_1' = \alpha_1 - \frac{a_1 \cdot b_1}{[1-(n(\omega_1)\cdot n(\omega_2))^2]^{1/2}}\, \alpha_d$$

$$\beta_1 = \frac{v_1 v_2}{[1-(n(\omega_1)\cdot n(\omega_2))^2]^{1/2}}\, \alpha_d$$

for (α_1', β_1). The Jacobian is

$$\left|\frac{\partial(\alpha_1', \alpha_2,..., \alpha_{d-1}, \beta_1,..., \beta_{d-1})}{\partial(\alpha_1, \alpha_2,..., \alpha_d, \beta_2,..., \beta_{d-1})}\right| = \left| \det \begin{pmatrix} 1 & -\dfrac{a_1 \cdot b_1}{[1-(n(\omega_1)\cdot n(\omega_2))^2]^{1/2}} \\ 0 & \dfrac{v_1 v_2}{[1-(n(\omega_1)\cdot n(\omega_2))^2]^{1/2}} \end{pmatrix} \right|$$

$$= \frac{v_1 v_2}{[1-(n(\omega_1)\cdot n(\omega_2))^2]^{1/2}} \leqslant \eta^{\beta - 1}$$

Define

$$\rho_3 = e(v_1 \mathbf{p}(0, \omega_1) + v_2 \mathbf{p}(0, \omega_2) + \mathbf{q})$$

viewed as a function of $\alpha_1,..., \alpha_d, \beta_2,..., \beta_{d-1}$. Note that for $1 \leqslant i \leqslant d-1$

$$\left.\frac{\partial \rho_3}{\partial \alpha_i}\right|_0 = v_1\, \nabla e(v_1 \mathbf{p}(0, \bar{\omega}_1) + v_2 \mathbf{p}(0, \bar{\omega}_2) + \mathbf{q}) \cdot a_i$$

and

$$\left.\frac{\partial \rho_3}{\partial \alpha_d}\right|_0 = v_1\, \nabla e(v_1 \mathbf{p}(0, \bar{\omega}_1) + v_2 \mathbf{p}(0, \bar{\omega}_2) + \mathbf{q}) \cdot \frac{-a_1 \cdot b_1}{[1-(n(\omega_1)\cdot n(\omega_2))^2]^{1/2}}\, a_1$$

$$+ v_2\, \nabla e(v_1 \mathbf{p}(0, \bar{\omega}_1) + v_2 \mathbf{p}(0, \bar{\omega}_2) + \mathbf{q}) \cdot \frac{v_1 v_2}{[1-(n(\omega_1)\cdot n(\omega_2))^2]^{1/2}}\, b_1$$

$$= v_1\, \nabla e(v_1 \mathbf{p}(0, \bar{\omega}_1) + v_2 \mathbf{p}(0, \omega_2) + \mathbf{q}) \cdot a_d$$

Because $a_1,..., a_d$ is an orthonormal basis, there is a j such that

$$\left|\frac{\partial\rho_3}{\partial\alpha_j}\right|\Big|_0 \geqslant \frac{1}{\sqrt{d}}\,|\nabla e(v_1\mathbf{p}(0,\bar{\omega}_1)+v_2\mathbf{p}(0,\omega_2)+\mathbf{q})| \geqslant \frac{1}{\sqrt{d}}\,g_0$$

and $|\partial\rho_3/\partial\alpha_j| \geqslant g_0/(2\sqrt{d})$ in a neighborhood of the origin. Make a final change of variables replacing α_j by ρ_3. The Jacobian for the composite change of variables from (ω_1, ω_2) to

$$(\alpha_1', \alpha_2,..., \alpha_{d-1}, \beta_1,..., \beta_{d-1})$$

to

$$(\alpha_1,..., \alpha_d, \beta_2,..., \beta_{d-1})$$

to

$$((\alpha_i)_{1\leqslant i\leqslant d, i\neq j}, (\beta_i)_{2\leqslant i\leqslant d-1}, \rho_3)$$

is bounded by const $\cdot\, \eta^{\beta-1}$. Covering $S\times S$ by a finite number of such coordinate patches, we have

$$T(\varepsilon) \leqslant \text{const}\cdot\eta^{\beta-1}\int_{-\text{const}\cdot\eta}^{\text{const}\cdot\eta} d\rho_3 \leqslant \text{const}\cdot\eta^{\beta} \tag{A.10}$$

The contribution from the set of exceptional momenta $\mathscr{E}$ is bounded using Assumption A3: fix $\omega_1 \in S$, let $\mathscr{D}(\omega_1)$ be as in (1.31), and let

$$\mathscr{E}_{\omega_1} = \{\omega\in S\colon [1-(n(\omega_1)\cdot n(\omega_2))^2]^{1/2} < \eta^{1-\beta}\} \tag{A.11}$$

Then by A3(ii), $\mathscr{E}_{\omega_i} \subset U_r(\mathscr{D}(\omega_1))$ with $r = (\eta^{1-\beta}/Z_1)^{1/\rho}$ (this ρ is now that from A3, not the "radial" coordinate), so by A3(i),

$$E(\varepsilon) \leqslant \int_S d\omega_1 \int_{U_r(\mathscr{E}_{\omega_1})} d\omega_2 \leqslant (Z_0(Z_1)^{-1/\rho})\,\eta^{\kappa(1-\beta)/\rho} \tag{A.12}$$

The optimal bound is when $\kappa(1-\beta)/\rho = \beta$, that is, $\beta = \kappa/(\kappa+\rho)$. ∎

Lemma A.2. Let $\mathscr{A} = \mathscr{A}_2(\sigma, \mathscr{N}, g_0, g_2, g_3)$ be as in (1.53). Then $\mathscr{A}$ is open, and ρ, κ, Z_0, Z_1, and thus C_{vol} can be chosen uniformly on $\mathscr{A}$, i.e., (1.33) holds with the same ε and C_{vol} for all $e\in\mathscr{A}$.

Proof. It is obvious by definition of $\mathscr{A}$ that it is an open set. Let $\omega\in S = S(e)$. Since $|n(\omega)| = 1$, $dn(\omega)\,h$ is orthogonal to $n(\omega)\in\mathbb{R}^d$ for all h in the tangent space at ω. Now, $\mathscr{D}(\omega)$, defined in (1.31), is the zero set of

$\phi_\omega\colon S \to \Omega^{(2)}(S)$, $\omega' \mapsto n(\omega') \wedge n(\omega)$. Here $d\phi_\omega(\omega') = dn(\omega') \wedge n(\omega)$, the mentioned orthogonality, and rank $dn \geqslant \sigma$ imply that $\mathscr{D}(\omega)$ is a C^{k-1}-submanifold of S of codimension $\geqslant \sigma$. It is now clear that Assumption A3 holds, with $\kappa = \sigma$, $\rho = 1$, and with Z_0 and Z_1 depending on the smallest eigenvalue (in absolute value), hence bounded by a function of g_3. Uniformity of C_{vol} on $\mathscr{A}_2(\sigma, \mathscr{N}, g_0, g_2, g_3)$ follows from (A.7), that of ε from Proposition 1.1. ▮

APPENDIX B. THE ONE-FERMION PROBLEM

Let $d \geqslant 2$, Γ be a lattice of maximal rank in $\mathbb{R}^d$, and

$$\Gamma^{\#} = \{ b \in \mathbb{R}^d \mid \langle b, \gamma \rangle \in 2\pi\mathbb{Z} \text{ for all } \gamma \in \Gamma \} \tag{B.1}$$

be its dual. Let $q(\mathbf{x})$ be a smooth potential that is periodic with respect to Γ. Then the Schrödinger operator $-(1/2m)\,\Delta + q(\mathbf{x})$ commutes with the unitary lattice translation operators

$$(T^\gamma \phi)(\mathbf{x}) = \phi(\mathbf{x} + \gamma), \qquad \gamma \in \Gamma \tag{B.2}$$

so that the spectrum of $-(1/2m)\,\Delta + q(\mathbf{x})$ is the union over $\kappa \in \mathbb{R}^d/\Gamma^{\#}$ of the spectra of the boundary value problem

$$\begin{gathered} \left(-\frac{1}{2m}\Delta + q \right) \phi = \lambda \phi \\ \phi(\mathbf{x} + \gamma) = e^{i\langle \kappa, \gamma \rangle} \phi(x) \qquad \forall \gamma \in \Gamma \end{gathered} \tag{B.3}$$

Label the eigenvalues of this problem, in increasing order, $\varepsilon_\nu(\kappa)$, $\nu \in \mathbb{N}$. Denote the corresponding eigenfunctions $\phi_{\kappa,\nu}(\mathbf{x})$ and normalize them by the condition that

$$\int_{\mathbb{R}^d/\Gamma} d\mathbf{x}\, |\phi_{\kappa,\nu}(\mathbf{x})|^2 = V_\Gamma := \mathrm{Vol}(\mathbb{R}^d/\Gamma) \tag{B.4}$$

This normalization is chosen so that when $q = 0$, $\{\phi_{\kappa,\nu} \mid \nu \in \mathbb{N}\} = \{ e^{i\langle \mathbf{k}, \mathbf{x}\rangle} \mid \mathbf{k} \in \kappa + \Gamma^{\#} \}$, and $\varepsilon_\nu(\kappa) = (1/2m)\,\mathbf{k}^2$. The boundary value problem

$$\begin{gathered} \left(-\frac{1}{2m}\Delta + q \right) \phi = \lambda \phi \\ \phi(\mathbf{x} + \gamma) = e^{i\langle \kappa, \gamma \rangle} \phi(x) \qquad \forall \gamma \in \Gamma \end{gathered} \tag{B.5}$$

is unitarily equivalent to

$$\left(\frac{1}{2m}(-i\nabla+\kappa)^2+q\right)u=\lambda u \qquad u(\mathbf{x}+\gamma)=u(x) \qquad \forall\gamma\in\Gamma \tag{B.6}$$

As $(-i\nabla+\kappa)^2$ is an analytic relatively bounded perturbation of $-\Delta$, the eigenvalues $\varepsilon_\nu(\kappa)$ and eigenfunctions $\phi_{\kappa,\nu}(x)$ depend analytically on κ at every κ for which $\varepsilon_\nu(\kappa)$ is a simple eigenvalue. The Fermi surface

$$\mathscr{S}_\mu=\{\kappa\mid\exists\nu \text{ such that } \varepsilon_\nu(\kappa)=\mu\} \tag{B.7}$$

for chemical potential μ is smooth at every κ for which

$$\varepsilon_\nu(\kappa)=\mu\Rightarrow \text{(a) } \varepsilon_\nu(\kappa) \text{ is a simple eigenvalue} \qquad \text{(b) } \nabla\varepsilon_\nu(\kappa)\neq 0 \tag{B.8}$$

Since $\nabla\varepsilon_\nu(\kappa)=0$ is a system of d equations in d unknowns κ, condition (b) is generically violated only at isolated points in $\mathbb{R}^d/\Gamma^{\#}$. In this paper, we exclude it by Assumption A2. We also restrict consideration to one band since for bands separated by a gap, the band index plays no interesting role.

The free two-point Schwinger function is

$$\begin{aligned}
C(\xi,\xi') &= -\langle\Phi_0, T[\psi(\xi)\,\bar\psi(\xi')]\,\Phi_0\rangle\\
&= -\frac{1}{(V_\Gamma L^d)^2}\sum_{\mathbf{k},\mathbf{k}'}\phi_{\mathbf{k}}(\xi)\,\bar\phi_{\mathbf{k}'}(\xi')\,e^{-(\varepsilon_\nu(\kappa)-\mu)\tau}e^{(\varepsilon_{\nu'}(\kappa')-\mu)\tau'}\\
&\quad\times\langle\Phi_0, T[a_{\mathbf{k},\sigma}a^\dagger_{\mathbf{k}',\sigma'}]\,\Phi_0\rangle\\
&= -\frac{1}{V_\Gamma L^d}\sum_{\mathbf{k},\mathbf{k}'}\phi_{\mathbf{k}}(\xi)\,\bar\phi_{\mathbf{k}'}(\xi')\,e^{-(\varepsilon_\nu(\kappa)-\mu)\tau}e^{(\varepsilon_{\nu'}(\kappa')-\mu)\tau'}\delta_{\mathbf{k},\mathbf{k}'}\delta_{\sigma,\sigma'}\\
&\quad\times\begin{cases}-\chi(\varepsilon_\nu(\mathbf{k})<\mu), & \tau\leqslant\tau'\\ \chi(\varepsilon_\nu(\mathbf{k})>\mu), & \tau>\tau'\end{cases}\\
&= \delta_{\sigma,\sigma'}\frac{1}{V_\Gamma L^d}\sum_{\mathbf{k}}\phi_{\mathbf{k}}(\xi)\,\bar\phi_{\mathbf{k}}(\xi')\,e^{-(\varepsilon_\nu(\kappa)-\mu)(\tau-\tau')}\\
&\quad\times\begin{cases}\chi(\varepsilon_\nu(\mathbf{k})<\mu), & \tau\leqslant\tau'\\ -\chi(\varepsilon_\nu(\mathbf{k})>\mu), & \tau>\tau'\end{cases}\\
&= \delta_{\sigma,\sigma'}\frac{1}{V_\Gamma L^d}\sum_{\mathbf{k}}\int\frac{dk_0}{2\pi}\,\phi_{\mathbf{k}}(\xi)\,\bar\phi_{\mathbf{k}}(\xi')\,e^{-ik_0(\tau-\tau')}\frac{1}{ik_0-e_\nu(\kappa)}
\end{aligned} \tag{B.9}$$

where

$$e_\nu(\kappa) = e_\nu(\kappa) - \mu \tag{B.10}$$

and for $\tau = \tau'$ the limit $\tau - \tau' \nearrow 0$ is implied. In the infinite-volume limit

$$C(\xi, \xi') = \delta_{\sigma,\sigma'} \sum_\nu \int_{\mathbb{R} \times \mathbb{R}^d/\Gamma^\#} \frac{dk_0}{2\pi} \frac{d\kappa}{(2\pi)^d} \phi_{\mathbf{k}}(\xi)\, \bar{\phi}_{\mathbf{k}}(\xi')\, e^{-ik_0(\tau - \tau')} \frac{1}{ik_0 - e_\nu(\kappa)} \tag{B.11}$$

Since we consider only one band, we drop ν and set $\kappa = \mathbf{k}$.

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